

An introduction to Seiberg-Witten theory on closed 3-manifolds

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Abstract

This is a version of the author's diploma thesis written at the University of Cologne in 2002/03. The topic is the construction of Seiberg-Witten invariants of closed 3-manifolds. In analogy to the four dimensional case, the structure of the moduli space is investigated. The Seiberg-Witten invariants are defined and their behaviour under deformation of the Riemannian metric is analyzed.

Since it is essentially an exposition of results which were already known during the time of writing, the thesis has not been published. In particular, the author does not claim any originality concerning the results. Moreover, new developments of the theory are not included. However, the detailed account—together with the appendices on the required functional analytic and geometric background—might be of interest for people starting to work in the area of gauge field theory.

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Introduction

1 Historical remarks and background

Seiberg-Witten theory is a gauge theoretical approach to low dimensional topology. By making use of Riemannian geometry and the analysis of partial differential equations, one finds smooth invariants of three or four dimensional manifolds, thus entering the area of differential topology.

1.1 Low dimensional topology

There has been a long standing interest in understanding the structure of manifolds. In this process, the dimensions three and four have always played a prominent role since an easy classification as in the two dimensional case is not possible. On the other hand, the very efficient topological tools which have proved useful in the theory of higher dimensional manifolds cannot be applied in the dimensions three and four. Among many other results alluding to these peculiarities are, for example, the discovery of exotic structures on \mathbb{R}^4 .

At the beginning of the 1980s, the work of M. Freedman gave a new insight in the topological classification of simply connected compact 4-manifolds via their intersection forms. About the same time, S.K. Donaldson succeeded in establishing criteria how the intersection form can prevent a topological 4-manifold from being smoothable.

Mathematical gauge theory. The main idea of Donaldson's work is to study solutions of the anti self-dual instanton equations—a set of partial differential equations arising from Yang-Mills theory, which describes elementary particles in physics. By making use of gauge symmetries, one defines the so-called *moduli space* whose structure reflects much of the underlying manifold's topology. It turns out that generically, the moduli space is a finite dimensional manifold with boundary except at a finite number of singular points occurring at solutions having too much gauge symmetry. By means of

establishing a relation between these singularities and the intersection form, Donaldson's famous Theorem A excludes certain 4-manifolds from admitting a differentiable structure.

Subsequent work—part of which culminated in the definition of Donaldson's polynomial invariants—has emphasized the fruitfulness of the gauge theoretical approach to low dimensional topology. As self-contained introductions to this development—written by some of the major participants—the books of Donaldson & Kronheimer [15] and Freed & Uhlenbeck [17] are highly recommended.

Gauge theory on 3-manifolds. The basic idea of gauge theory is not restricted to the four dimensional setting. It thus was soon applied to 3-manifolds as well. However, since the anti self-dual equations cannot be formulated on a manifold of this dimension, the viewpoint had to be altered a little. The critical points of the so-called *Chern-Simons function* were found to be a promising substitute.¹ An important concept of this work—particularly due to C.H. Taubes [52] and A. Floer [16]—is to interpret the Chern-Simons function as a Morse function on the space of all gauge fields modulo the action of the group of gauge transformations. An invariant is then defined in the same manner as in finite dimensional Morse theory: There, the Euler characteristic of a manifold can be computed as the signed count of Morse indices. The major problems connected with this idea, namely the question of how to deal with degenerate critical points and how to generalize the notion of the Morse index to an infinite dimensional setting, has successfully been solved by Taubes and Floer. The invariant obtained by using their approach turned out to equal the topologically defined Casson invariant.

In fact, Floer's work went much beyond that. He generalized the concept of a Morse complex and constructed cohomology groups associated to the Chern-Simons function which yield even more refined invariants. A recent monograph by Donaldson [14] gives a detailed exposition of the so-called *Floer homology groups* in the gauge theoretical context.

Seiberg-Witten theory. While the anti self-dual instanton equations of pure Yang-Mills theory are easily written down, the involved calculations are complicated due to the non-abelian nature of the symmetry group. In 1994, new impulses in mathematical gauge theory came from E. Witten's famous article [60]. He announced that a system of partial differential equations—the *monopole equations* which arose in his joint work with N. Seiberg—should in some sense be equivalent to the anti self-dual instanton equations. How-

¹At least on homology spheres.

ever, the Seiberg-Witten equations have an abelian gauge symmetry and are therefore easier to be dealt with from an analytical point of view. For example, Witten proved that the corresponding moduli space is always compact so that smooth invariants can be extracted in a much easier way than in instanton theory.

In the subsequent months, many of the results obtained via Donaldson theory could be reproved by making use of what was soon called *Seiberg-Witten theory*. For example, P.B. Kronheimer and T. Mrowka established a remarkably simple proof of the Thom conjecture [28]. Moreover—as Witten pointed out in his original paper—the structure of the Seiberg-Witten equations simplify considerably when formulated on Kähler surfaces so that the new theory had an instantaneous impact on complex geometry. Soon, C.H. Taubes managed in a series of papers—starting with [53]—to establish deep relations to invariants of symplectic 4-manifolds. Beginning with the work of C. LeBrun [30], Seiberg-Witten theory was also found promising in answering unsolved questions in Riemannian geometry.

Although the pace in which new results were obtained decreased after a while, Seiberg-Witten theory has become an important tool for studying 4-manifolds. Nowadays there are not only many survey articles reviewing the dawn of Seiberg-Witten theory (e.g. Donaldson [13], Kronheimer [27] and recently K. Iga [22]) but also monographs giving a more detailed exposition of the theory (e.g. J.W. Morgan [42] and J.D. Moore [41]). The book of L.I. Nicolaescu [45] is perhaps the most extensive introduction to the four dimensional theory which has appeared until now. M. Marcolli’s textbook [35] provides a remarkable selection of excellent references for any aspect of Seiberg-Witten theory—including the physical background.

1.2 Seiberg-Witten theory on 3-manifolds

It was soon realized by P.B. Kronheimer and T. Mrowka in [28] that the four dimensional theory can be carried over to 3-manifolds if the Seiberg-Witten equations are studied on a manifold $M \times S^1$, where M is a compact 3-manifold. Subsequently, much concentration was focussed on studying the new theory from a three dimensional point of view as well.

Kronheimer and Mrowka found out that as in instanton theory, the partial differential equations obtained for 3-manifolds have a natural interpretation as the gradient flow equations of a Chern-Simons-like function. Moreover, the Seiberg-Witten moduli space of a Riemannian 3-manifold is always compact. Due to the variational aspects of the theory, it turns out that—up to a generic perturbation—the moduli space consists of isolated points. Applying Taubes’ and Floer’s ideas from instanton theory it is thus possible to define

the signed count of monopoles, which is again reminiscent of expressing a finite dimensional manifold's Euler characteristic in terms of a Morse function's critical points. The number obtained in this way is then expected to be independent of the chosen metric and the perturbation term.

It was proved by Y. Lim in [32], for example, that this is indeed true for manifolds with first Betti number $b_1 > 1$, whereas in the case $b_1 = 1$ the number depends on a certain cohomological datum encoded in the perturbation term. In [38], G. Meng and C.H. Taubes exposed a relationship between a version of the Seiberg-Witten invariant and the Milnor torsion invariant for manifolds with $b_1 \geq 1$.

For rational homology spheres, however, one finds a severe dependence on the underlying Riemannian structure and the perturbation term. Nevertheless, there is the possibility of adding a counter term—a certain combination of η -invariants—to the signed count of monopoles so that the sum obtained in this way has the desired invariance properties. It was conjectured by Kronheimer and later independently proved by W. Chen [11] and Y.Lim [31] that in the case of an integer homology sphere, the number obtained in this way equals the Casson invariant. Moreover, for rational homology spheres, there is a relation to the so-called Casson-Walker invariant (cf. M. Marcolli & B.L. Wang [37] and L.I. Nicolaescu [46]).

Seiberg-Witten-Floer homology. Very soon after the appearance of the new theory, Donaldson pointed out in [13] that Floer's construction of a Morse complex associated to the Chern-Simons function should carry over to Seiberg-Witten theory as well. A derivation of three dimensional theory from the four dimensional case, pointing out the physical background and the connection to topological quantum field theory, was performed by A.L. Carey et al. in [10] and made establishing a Seiberg-Witten Floer homology even more demanding.

For manifolds with non-vanishing first Betti number, it was soon accomplished by M. Marcolli in [34] to build-up the Morse complex and prove its topological invariance. About the same time, B.L. Wang [56] exposed a severe dependence on the metric in the case of homology spheres. Subsequently, many authors began approaching the problem of defining a unified Seiberg-Witten-Floer homology for all 3-manifolds and much of this task seems to be solved nowadays (cf. Marcolli & Wang [36] and K. Iga [21]).

2 Organization of this thesis

The goal of this thesis is to present a detailed and largely self-contained construction of Seiberg-Witten invariants on closed 3-manifolds. We take a purely gauge theoretical point of view and shall not attempt to expose the relations to other topological invariants to which we have alluded above. In this sense, we restrict ourselves to only one—though the major—aspect of three dimensional Seiberg-Witten theory. With a view towards Seiberg-Witten-Floer theory, we emphasize the Morse theoretical aspects of the constructions but again, a more detailed integration of this far reaching subject is beyond the scope of this thesis.

The organization of the chapters is as follows:

- Chapter I establishes the gauge theoretical set-up in which the three dimensional Seiberg-Witten equations are formulated.
- Chapter II investigates the topological structure of the moduli space in analogy to the four dimensional case. Understanding the local structure of the moduli space will then make it possible to define the signed count of monopoles in the same way as it is performed in Taubes’ work on instanton theory.
- Chapter III is devoted to the analysis of how the signed count of monopoles depends on the metric. Following the work of Lim [32], Chen [11] and Nicolaescu [43], we shall establish the main theorems of this thesis, which prove invariance for manifolds with $b_1 > 1$, provide a “wall-crossing” formula in the case $b_1 = 1$, and exhibit the severe dependence on the metric in the case of rational homology spheres.

Since gauge theory requires nontrivial geometrical and functional analytic constructions, we append short summaries of the material we need:

- Appendix A contains a survey of the functional analytic aspects of nonlinear elliptic equations on compact manifolds.
- In Appendix B, we present a version of the determinant line bundle over the space of Fredholm operators which is needed in gauge theory to equip moduli spaces with an orientation.
- In Appendix C, the notion of spectral flow is recalled, which we shall need to exhibit a geometrical interpretation of the orientation of the moduli space as the signed count of critical points.

- The material needed to understand the geometrical set-up of Seiberg-Witten theory is presented in Appendix D.

Even though familiarity with most of these constructions is assumed, the reader is advised to browse through the appendices since it is there, where most notations are fixed.

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Chapter I

Seiberg-Witten Monopoles

The Seiberg-Witten equations are formulated within a framework arising from spin geometry. They are a set of partial differential equations involving a spinor field—the “matter field”—and a connection on a certain Hermitian line bundle—the “gauge field”.

In this chapter we describe the special set-up arising in the three dimensional context. We follow the notation of Appendix D where an exposition of spin^c manifolds is given.

To describe the interrelation between the curvature of the gauge field—the “field strength”—and the spinor, we have to perform some purely linear algebraic constructions. This is the content of Section 1. Having done so, we shall formulate the Seiberg-Witten equations in Section 2. With a view towards the Morse theoretical approach to three dimensional Seiberg-Witten theory, we then interpret solutions to these equations as the critical points of a Chern-Simons-like functional. We shall also see how the Seiberg-Witten equations fit into the context of elliptic equations.

1 Algebraic preliminaries

Spin representation in dimension three. Let (V, g) be an oriented three dimensional Euclidean vector space. The complex Clifford algebra $\text{Cl}^c(V)$ is isomorphic to $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ denotes the ring of (2×2) -matrices. If (e_1, e_2, e_3) is an oriented orthonormal basis of V , then this isomorphism has an explicit description, which is given by its action on this basis via

$$\text{Cl}^c(V) \rightarrow M_2(\mathbb{C}) \oplus M_2(\mathbb{C}), \quad e_j \mapsto \begin{pmatrix} i\sigma_j & 0 \\ 0 & -i\sigma_j \end{pmatrix}.$$

Here, σ_j denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The two non-isomorphic irreducible representations of $\text{Cl}^c(V)$ on $\Delta := \mathbb{C}^2$ are given by $e_j \mapsto i\sigma_j$ and $e_j \mapsto -i\sigma_j$ respectively. We require that the complex volume element $\omega^c = -e_1 e_2 e_3$ (cf. (D, 1)) acts as the identity on Δ , i.e., we fix the latter representation¹. In particular, Clifford multiplication takes the following form:

$$c : V \rightarrow \text{End}(\Delta), \quad c(e_j) = -i\sigma_j.$$

It is skew Hermitian with respect to the standard metric $\langle \cdot, \cdot \rangle$ on Δ .

The quadratic map. Let $\psi \in \Delta$ be a spinor. We define a linear map $q(\psi) : V \rightarrow \mathbb{C}$ by letting $q(\psi)(v) := -\frac{1}{2}\langle c(v)\psi, \psi \rangle$. With respect to an orthonormal basis:

$$\boxed{q(\psi) = -\frac{1}{2}\langle c(e_j)\psi, \psi \rangle e^j}, \quad (\text{I, 1})$$

where (e^1, e^2, e^3) denotes the dual basis and we take the sum² over all j . Since $c(e_j)$ is skew Hermitian, $q(\psi)$ is a purely imaginary valued co-vector. We thus obtain a quadratic map

$$q : \Delta \rightarrow V^* \otimes_{\mathbb{R}} i\mathbb{R} =: iV^*.$$

Polarization gives the associated \mathbb{R} -bilinear map

$$q(\psi, \varphi) = \frac{1}{4}(q(\psi + \varphi) - q(\psi - \varphi)) = -\frac{1}{2}i \text{Im} \langle c(e_j)\psi, \varphi \rangle e^j. \quad (\text{I, 2})$$

Clifford multiplication extends to iV^* via action on the co-vector part, i.e.,

$$c(i\alpha)\psi := ic(v_\alpha)\psi,$$

where $\psi \in \Delta$, $\alpha \in V^*$, and v_α denotes the metric dual of α . Observe that Clifford multiplication with imaginary valued co-vectors is Hermitian and trace-free. In fact, we have

Lemma (I, 1.1). *Clifford multiplication is an isomorphism of \mathbb{R} vector spaces*

$$c : iV^* \rightarrow \{T \in \text{End}(\Delta) \mid T \text{ Hermitian, } \text{Tr } T = 0\}.$$

¹There is some ambiguity in the literature but most authors consider this representation as the standard one.

²When using coordinates we shall always use the Einstein convention, i.e., we sum over all indices appearing twice.

Proof. The map is injective because $c(i\alpha)^2 = -c(v_\alpha)^2 = |v_\alpha|^2 \text{id}$. Since the \mathbb{R} vector space of trace-free and Hermitian endomorphisms on Δ is three dimensional, the result follows. \square

Remark. The trace-free and Hermitian endomorphism given by Clifford multiplication with $q(\psi)$ is given by

$$\boxed{c(q(\psi)) = \psi^* \otimes \psi - \frac{1}{2}|\psi|^2 \text{id}}, \quad (\text{I, 3})$$

which means³ that $c(q(\psi))\varphi = \langle \varphi, \psi \rangle \psi - \frac{1}{2}|\psi|^2 \varphi$ for any spinor φ . With respect to any unitary basis of Δ , this endomorphism has the matrix description

$$c(q(\psi)) = \frac{1}{2} \begin{pmatrix} |\alpha|^2 - |\beta|^2 & 2\alpha\bar{\beta} \\ 2\bar{\alpha}\beta & |\beta|^2 - |\alpha|^2 \end{pmatrix}, \quad \psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

As we shall not use this description, we skip the proof of (I, 3). With the explicit representation given by the Pauli matrices the involved computations are rather simple. The description (I, 3) is more satisfactory than definition (I, 1) because it is invariant of any explicit representation and can easily be carried over to other dimensions. However, the definition we gave is much more convenient for explicit calculations.

We endow the \mathbb{R} vector space iV^* with the scalar product induced by g , i.e., we let

$$\langle i\alpha, i\beta \rangle_g := g(v_\alpha, v_\beta),$$

where $\alpha, \beta \in V^*$ with metric duals v_α, v_β . Later on, we shall drop the subscript g for notational convenience. However, for the time being, we keep it to distinguish $\langle \cdot, \cdot \rangle_g$ from the Hermitian scalar product $\langle \cdot, \cdot \rangle$ on Δ .

Proposition (I, 1.2). *For all spinors ψ and φ the following holds:*

- (i) $\langle a, q(\psi, \varphi) \rangle_g = \frac{1}{2} \text{Re} \langle c(a)\psi, \varphi \rangle$ for any $a \in iV^*$.
- (ii) $|q(\psi, \varphi)|_g^2 = \frac{1}{4} \left(|\psi|^2 |\varphi|^2 - (\text{Re} \langle i\psi, \varphi \rangle)^2 \right)$. In particular, $|q(\psi)|_g = \frac{1}{2} |\psi|^2$.
- (iii) If $\psi \neq 0$, then $\varphi \in \ker q(\psi, \cdot)$ if and only if φ is a multiple of ψ by an imaginary number.

³Observe that we use the convention that Hermitian metrics are complex linear in the first entry.

Proof. Let (e_1, e_2, e_3) be an oriented orthonormal basis of (V, g) .

(i) We write $a = i\alpha_j e^j$ with $\alpha_j \in \mathbb{R}$. Then

$$\begin{aligned} \langle a, q(\psi, \varphi) \rangle_g &= -\frac{1}{2} \left\langle i\alpha_j e^j, i \operatorname{Im} \langle c(e_k) \psi, \varphi \rangle e^k \right\rangle_g \\ &= -\frac{1}{2} \alpha_j \operatorname{Im} \langle c(e_k) \psi, \varphi \rangle \delta^{jk} = -\frac{1}{2} \operatorname{Im} \langle c(\alpha_j e^j) \psi, \varphi \rangle \\ &= -\frac{1}{2} \operatorname{Im} \langle -ic(i\alpha_j e^j) \psi, \varphi \rangle = \frac{1}{2} \operatorname{Re} \langle c(a) \psi, \varphi \rangle. \end{aligned}$$

(ii) Without loss of generality, we may assume that $\psi \neq 0$. If we interpret Δ as a 4-dimensional Euclidean vector space with respect to the real scalar product $\operatorname{Re}\langle \cdot, \cdot \rangle$, then the elements $i\psi$, $c(e_1)i\psi$, $c(e_2)i\psi$, $c(e_3)i\psi$ form an orthogonal basis of Δ . We thus have

$$|\psi|^2 |\varphi|^2 = (\operatorname{Re} \langle i\psi, \varphi \rangle)^2 + \sum_{j=1}^3 (\operatorname{Re} \langle c(e_j)i\psi, \varphi \rangle)^2.$$

This in mind, we conclude:

$$\begin{aligned} |q(\psi, \varphi)|_g^2 &= \frac{1}{4} \sum_{j=1}^3 (\operatorname{Im} \langle c(e_j) \psi, \varphi \rangle)^2 = \frac{1}{4} \sum_{j=1}^3 (\operatorname{Re} \langle c(e_j)i\psi, \varphi \rangle)^2 \\ &= \frac{1}{4} (|\varphi|^2 |\psi|^2 - (\operatorname{Re} \langle i\psi, \varphi \rangle)^2). \end{aligned}$$

(iii) According to part (ii), we have

$$q(\psi, \varphi) = 0 \iff |\psi| |\varphi| = |\operatorname{Re} \langle i\psi, \varphi \rangle|.$$

It thus follows from the Cauchy-Schwarz inequality that $q(\psi, \varphi) = 0$ if and only if φ is a real multiple of $i\psi$. \square

Hodge-star-operator and wedge product. We recall that there is an isometry on $\Lambda^\bullet V^*$,

$$*: \Lambda^k V^* \rightarrow \Lambda^{3-k} V^*, \quad k \in \{0, \dots, 3\},$$

uniquely characterized by the property that

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle dv_g, \quad \alpha \in \Lambda^k V^*, \beta \in \Lambda^{3-k} V^*.$$

Here, dv_g denotes the oriented volume element of V . The fact that V is three dimensional implies that

$$*^2 = \operatorname{id}_{\Lambda^\bullet V^*}.$$

We shall also need the Hodge-star-operator on $i\Lambda^\bullet V^*$. This is because $i\mathbb{R}$ is the Lie algebra of U_1 and in gauge theory, Lie algebra valued forms play a decisive role. We adapt the standard convention and extend $*$ complex linearly, i.e., we let $*(i\alpha) = i*\alpha$, where $\alpha \in \Lambda^\bullet V^*$. This is not to be confused with regarding $i\Lambda^\bullet V^*$ as a subspace of $\Lambda^\bullet V^* \otimes \mathbb{C}$ endowed with the complex anti-linear Hodge-star-operator of complex differential geometry.

Interpreting $i\mathbb{R}$ as a Lie algebra, there is a canonical way of defining a wedge product on $i\Lambda^\bullet V^*$. As $i\mathbb{R}$ is an abelian Lie algebra, the so defined product would, however, vanish. In contrast to the above, we will therefore use the wedge product of $\Lambda^\bullet V^* \otimes \mathbb{C}$. This gives the possibility to form the wedge product of an imaginary valued co-vector with a real valued one. Note that for $\alpha, \beta \in \Lambda^\bullet V^*$

$$i\alpha \wedge i\beta = -\alpha \wedge \beta. \quad (\text{I, 4})$$

As a consequence of our conventions, $a \in i\Lambda^\bullet V^*$ satisfies

$$a \wedge *a = -\langle a, a \rangle dv_g.$$

2 The Seiberg-Witten equations

Let (M, g) be a closed,⁴ oriented Riemannian 3-manifold. According to Proposition (D, 2.11), M admits a spin^c structure. Fixing $\sigma \in \text{spin}^c(M)$, we let $L(\sigma)$ denote the associated Hermitian line bundle, and let $S = S(\sigma)$ be the fundamental spinor bundle over M associated to the Spin^c -bundle $P_{\text{Spin}^c}(\sigma)$ via the representation chosen in Section 1. Then $S(\sigma)$ is a Hermitian vector bundle of rank 2 over M .

The quadratic map q extends to a morphism $C^\infty(M, S) \rightarrow i\Omega^1(M)$. For later purposes we establish a necessary condition for $q(\psi)$ to be co-closed.

Proposition (I, 2.1). *Let A be an arbitrary Hermitian connection on $L(\sigma)$, \mathcal{D}_A the associated spin^c Dirac operator. Then we have the (pointwise) identity*

$$d^*q(\psi) = i \text{Im} \langle \mathcal{D}_A \psi, \psi \rangle.$$

In particular, q is co-closed whenever ψ is a harmonic spinor.

Proof. At an arbitrary point x_0 , we consider a normal⁵ frame (e_1, e_2, e_3) with dual co-frame (e^1, e^2, e^3) . This implies that

$$\nabla_j^A(c(e_i)\psi)(x_0) = (c(e_i)\nabla_j^A\psi)(x_0).$$

⁴We use the convention that a closed manifold is a compact and connected manifold without boundary. Although the assumption about connectedness is usually not standard, we include it here for simplicity of notation.

⁵Recall that this means $(\nabla_i e_j)(x_0) = 0$.

Using that in the case at hand, $d^* = - * d *$, that A is Hermitian and that $c(e_i)$ is skew Hermitian, we find that at the point x_0

$$\begin{aligned}
 d^* q(\psi) &= -\frac{1}{2} d^* \langle c(e_i) \psi, \psi \rangle e^i = \frac{1}{2} * d * \langle c(e_i) \psi, \psi \rangle e^i \\
 &= \frac{1}{2} * \left(\langle c(e_i) \nabla_j^A \psi, \psi \rangle + \langle c(e_i) \psi, \nabla_j^A \psi \rangle \right) e^j \wedge * e^i \\
 &= \frac{1}{2} \left(\langle c(e_i) \nabla_i^A \psi, \psi \rangle - \langle \psi, c(e_i) \nabla_i^A \psi \rangle \right) * (e^i \wedge * e^i) \\
 &= \frac{1}{2} \left(\langle \mathcal{D}_A \psi, \psi \rangle - \langle \psi, \mathcal{D}_A \psi \rangle \right) * dv_g \\
 &= i \operatorname{Im} \langle \mathcal{D}_A \psi, \psi \rangle.
 \end{aligned}$$

Note that we have also employed the local description of \mathcal{D}_A (cf. (D, 25)) and the fact that $e^i \wedge * e^j = \delta^{ij} dv_g$. \square

Let $\mathcal{A}(\sigma)$ denote the affine space of Hermitian connections on $L(\sigma)$. We define the *configuration space* as

$$\mathcal{C} = \mathcal{C}(\sigma) := C^\infty(M, S(\sigma)) \times \mathcal{A}(\sigma).$$

Since $\mathcal{A}(\sigma)$ is an affine space modelled on $C^\infty(M, iT^*M)$, the configuration space is also an affine space which is modelled on $C^\infty(M, S \oplus iT^*M)$. To make formulæ clearer and notation shorter we define

$$E = E(\sigma) := S(\sigma) \oplus iT^*M.$$

The *group of gauge transformations* of a spin^c structure is (cf. Definition (D, 2.12))

$$\mathcal{G} := C^\infty(M, U_1).$$

Its natural operations on $C^\infty(M, S)$ and \mathcal{A} (cf. (D, 16) and (D, 24)) induce an action on the configuration space \mathcal{C} , given by

$$\begin{aligned}
 \mathcal{G} \times \mathcal{C}(\sigma) &\longrightarrow \mathcal{C}(\sigma), \\
 (\gamma, (\psi, A)) &\longmapsto (\gamma^{-1} \psi, A + 2\gamma^{-1} d\gamma).
 \end{aligned} \tag{I, 5}$$

We define the quotient of the configuration space with respect to the \mathcal{G} action by

$$\mathcal{B}(\sigma) := \mathcal{C}(\sigma) / \mathcal{G}.$$

The action of \mathcal{G} on \mathcal{C} lifts naturally to $C^\infty(M, E)$, the tangent space of \mathcal{C} at an arbitrary point (ψ, A) , via

$$\gamma \cdot (\varphi, a) := \left. \frac{d}{dt} \right|_{t=0} \gamma \cdot (\psi + t\varphi, A + ta) = (\gamma^{-1} \varphi, a). \tag{I, 6}$$

Definition (I, 2.2). $(\psi, A) \in \mathcal{C}(\sigma)$ is called *irreducible* if the stabilizer $\mathcal{G}_{(\psi, A)}$ of the \mathcal{G} action at the point (ψ, A) is trivial. Otherwise, it is called *reducible*. The subset of irreducible configurations is denoted by $\mathcal{C}^*(\sigma)$. and its quotient with respect to the \mathcal{G} action by $\mathcal{B}^*(\sigma)$.

Let $\gamma \neq 1$ lie in the stabilizer $\mathcal{G}_{(\psi, A)}$, i.e., $\gamma \cdot (\psi, A) = (\psi, A)$. Then $\gamma^{-1}\psi = \psi$ and $2\gamma^{-1}d\gamma = 0$. As M is connected, we deduce the following.

Lemma (I, 2.3). *A configuration (ψ, A) is reducible if and only if $\psi \equiv 0$. In this case, the stabilizer $\mathcal{G}_{(\psi, A)}$ consists of the constant maps $M \rightarrow \mathbb{U}_1$.*

The moduli space. We can now formulate the Seiberg-Witten equations.

Definition (I, 2.4). Let M be a closed, oriented Riemannian 3-manifold with spin^c structure σ . For each gauge field A , we let \mathcal{D}_A denote the spin^c Dirac operator associated to A , and $F_A \in i\Omega^2(M)$ the curvature 2-form of A . Then a configuration (ψ, A) is called a *Seiberg-Witten monopole* if it solves the equations

$$\boxed{\begin{array}{lcl} \mathcal{D}_A \psi & = & 0 \\ *F_A & = & \frac{1}{2}q(\psi) \end{array}}. \quad (\text{I, 7})$$

We can interpret Seiberg-Witten monopoles as the zeros of a vector field on \mathcal{C} . We define the *Seiberg-Witten map*

$$\text{SW} : \mathcal{C}(\sigma) \rightarrow C^\infty(M, E(\sigma)), \quad (\psi, A) \mapsto (\mathcal{D}_A \psi, \frac{1}{2}q(\psi) - *F_A).$$

Lemma (I, 2.5). *The map $\text{SW} : \mathcal{C} \rightarrow C^\infty(M, E)$ is equivariant with respect to the \mathcal{G} -actions (I, 5) and (I, 6) on \mathcal{C} and $C^\infty(M, E)$ respectively.*

Proof. Let (ψ, A) be an arbitrary configuration, and let $\gamma \in \mathcal{G}$. Then Lemma (D, 3.10) implies

$$\begin{aligned} \mathcal{D}_{(A+2\gamma^{-1}d\gamma)}(\gamma^{-1}\psi) &= \mathcal{D}_A(\gamma^{-1}\psi) + c(\gamma^{-1}d\gamma)\gamma^{-1}\psi \\ &= \gamma^{-1}\mathcal{D}_A\psi + c(d\gamma^{-1})\psi + c(\gamma^{-2}d\gamma)\psi = \gamma^{-1}\mathcal{D}_A\psi \end{aligned}$$

because $d\gamma^{-1} = -\gamma^{-2}d\gamma$. Furthermore, since $\gamma^{-1}d\gamma$ is closed,

$$\frac{1}{2}q(\gamma^{-1}\psi) - *F_{A+2\gamma^{-1}d\gamma} = \frac{1}{2}q(\gamma^{-1}\psi) - *(F_A + 2d(\gamma^{-1}d\gamma)) = \frac{1}{2}q(\psi) - *F_A.$$

Note that in the last equality we have used that $q(\lambda\psi) = q(\psi)$ for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. \square

In particular, the set of Seiberg-Witten monopoles is a \mathcal{G} -invariant subset of the configuration space \mathcal{C} and thus, the set

$$\mathcal{M}(\sigma) := \text{SW}^{-1}(0)/\mathcal{G} \subset \mathcal{B}(\sigma)$$

is a well-defined subset of the set of gauge equivalence classes. \mathcal{M} is called the *Seiberg-Witten moduli space* associated to the spin^c structure σ on the Riemannian manifold (M, g) . The subset of $\mathcal{M}(\sigma)$ given by gauge equivalence classes of irreducible monopoles is denoted by $\mathcal{M}^*(\sigma) \subset \mathcal{B}^*(\sigma)$.

Remark. The structure of the moduli space $\mathcal{M}(\sigma)$ depends heavily on the particular choice of g . Whenever we want to stress this dependence on the metric, we shall write $\mathcal{M}(\sigma; g)$ instead of $\mathcal{M}(\sigma)$.

Variational aspects of the Seiberg-Witten equations. We will now present a special feature of three dimensional Seiberg-Witten theory that does not carry over to the four dimensional case: We can interpret Seiberg-Witten monopoles as critical points of a Chern-Simons like functional. This was firstly observed by Kronheimer and Mrowka in [28].

Definition (I, 2.6). For every configuration (ψ, A) , we define the *Chern-Simons-Dirac functional*

$$\boxed{\text{csd}(\psi, A) := \frac{1}{2} \int_M \left(\langle \psi, \mathcal{D}_A \psi \rangle dv_g + (A - A_0) \wedge (F_A + F_{A_0}) \right)},$$

where A_0 is an arbitrary fixed connection on $L(\sigma)$. Observe that csd is \mathbb{R} -valued since \mathcal{D}_A is formally self-adjoint.

Remark. Note that the definition of the Chern-Simons-Dirac functional depends on the choice of A_0 so that we should write csd_{A_0} . However, if A_1 is another fixed connection on $L(\sigma)$, then a short calculation shows that

$$\text{csd}_{A_1} - \text{csd}_{A_0} = - \int_M F_{A_1} \wedge (A_1 - A_0),$$

i.e., the value of csd is well-defined up to an additive constant. Since we are only interested in critical points of csd we shall not bother stressing the dependence on A_0 .

We endow the \mathbb{R} vector space $C^\infty(M, E)$ with the pre Hilbert scalar product induced by the scalar products $\langle \cdot, \cdot \rangle$ on S and $\langle \cdot, \cdot \rangle_g$ on iT^*M , i.e.,

for $(\varphi, a), (\varphi', a') \in C^\infty(M, E)$ we let

$$\begin{aligned} ((\varphi, a), (\varphi', a'))_{L^2} &:= \operatorname{Re}(\varphi, \varphi')_{L^2} + (a, a')_{L^2} \\ &= \int_M \operatorname{Re}\langle \varphi, \varphi' \rangle dv_g + \int_M \langle a, a' \rangle_g dv_g. \end{aligned}$$

Proposition (I, 2.7). *The map $\operatorname{SW} : \mathcal{C} \rightarrow C^\infty(M, E)$ is the gradient of the Chern-Simons-Dirac functional with respect to the L^2 scalar product, i.e., for $(\psi, A) \in \mathcal{C}$ and $(\varphi, a) \in C^\infty(M, E)$:*

$$\left. \frac{d}{dt} \right|_{t=0} \operatorname{csd}(\psi + t\varphi, A + ta) = (\operatorname{SW}(\psi, A), (\varphi, a))_{L^2}.$$

The Hessian of csd at (ψ, A) is the formally self-adjoint first-order differential operator

$$F_{(\psi, A)} := D_{(\psi, A)} \operatorname{SW} : C^\infty(M, E) \rightarrow C^\infty(M, E),$$

given by

$$F_{(\psi, A)}(\varphi, a) = (\mathcal{D}_A \varphi + \frac{1}{2}c(a)\psi, q(\psi, \varphi) - *da). \quad (\text{I, 8})$$

Proof. Let $(\varphi, a) \in C^\infty(M, E)$. Then

$$\begin{aligned} &\left. \frac{d}{dt} \right|_{t=0} \operatorname{csd}(\psi + t\varphi, A + ta) \\ &= \left. \frac{d}{dt} \right|_{t=0} \frac{1}{2} \int_M \left(\operatorname{Re} \langle \psi + t\varphi, \mathcal{D}_A(\psi + t\varphi) + \frac{1}{2}c(ta)(\psi + t\varphi) \rangle dv_g \right. \\ &\quad \left. + (A + ta - A_0) \wedge (F_A + tda + F_{A_0}) \right) \\ &= \frac{1}{2} \int_M \left(\operatorname{Re} \langle \varphi, \mathcal{D}_A \psi \rangle dv_g + \operatorname{Re} \langle \psi, \mathcal{D}_A \varphi + \frac{1}{2}c(a)\psi \rangle dv_g \right. \\ &\quad \left. + a \wedge (F_A + F_{A_0}) + (A - A_0) \wedge da \right) \\ &= \int_M \operatorname{Re} \langle \varphi, \mathcal{D}_A \psi \rangle dv_g + \frac{1}{2} \int_M \operatorname{Re} \langle \psi, \frac{1}{2}c(a)\psi \rangle dv_g \\ &\quad - \frac{1}{2} \int_M \left(\langle a, *(F_A + F_{A_0}) \rangle + \langle A - A_0, *da \rangle \right) dv_g, \end{aligned}$$

where we have employed formal self-adjointness of \mathcal{D}_A and formula (I, 4) in the last line. Formal self-adjointness of $*d$ on 1-forms and Proposition (I, 1.2) show that this equals

$$\begin{aligned} &\operatorname{Re} (\mathcal{D}_A \psi, \varphi)_{L^2} + (\frac{1}{2}q(\psi), a)_{L^2} - \frac{1}{2} \left(*(F_A + F_{A_0}) + *d(A - A_0), a \right)_{L^2} \\ &= \operatorname{Re} (\mathcal{D}_A \psi, \varphi)_{L^2} + (\frac{1}{2}q(\psi) - *F_A, a)_{L^2} \\ &= (\operatorname{SW}(\psi, A), (\varphi, a))_{L^2}. \end{aligned}$$

The Hessian $F_{(\psi, A)}$ at (ψ, A) is computed as follows:

$$\begin{aligned} D_{(\psi, A)} \text{SW}(\varphi, a) &= \left. \frac{d}{dt} \right|_{t=0} \text{SW}(\psi + t\varphi, A + ta) \\ &= \left. \frac{d}{dt} \right|_{t=0} (\mathcal{D}_{A+ta}(\psi + t\varphi), \tfrac{1}{2}q(\psi + t\varphi) - *F_{A+ta}) \\ &= (\mathcal{D}_A\varphi + \tfrac{1}{2}c(a)\psi, q(\psi, \varphi) - *da). \end{aligned}$$

Observe that we have used that the differential of q at the point ψ is given by

$$D_\psi q(\varphi) = 2q(\psi, \varphi).$$

As it is the Hessian of csd , formal self-adjointness of $F_{(\psi, A)}$ is immediate. \square

Our main interest lies in gauge equivalence classes of critical points of the Chern-Simons-Dirac functional. However, one major observation is that $\text{csd} : \mathcal{C} \rightarrow \mathbb{R}$ is *not* gauge invariant. In fact,

$$\text{csd}(\gamma \cdot (\psi, A)) = \text{csd}(\psi, A) - 8\pi^2 \int_M \left[\frac{1}{2\pi i} \gamma^{-1} d\gamma \right] \wedge c(\sigma), \quad (\text{I, 9})$$

where $c(\sigma)$ denotes the canonical class of the spin^c structure σ (cf. Definition (D, 2.3)).

Proof. Since $d(\gamma^{-1}d\gamma) = 0$ and $d\gamma^{-1} = -\gamma^{-2}d\gamma$ we have

$$\begin{aligned} \text{csd}(\gamma \cdot (\psi, A)) &= \frac{1}{2} \int_M \left(\left\langle \gamma^{-1}\psi, \mathcal{D}_A(\gamma^{-1}\psi) + \tfrac{1}{2}c(2\gamma^{-1}d\gamma)(\gamma^{-1}\psi) \right\rangle dv_g \right. \\ &\quad \left. + (A + 2\gamma^{-1}d\gamma - A_0) \wedge (F_A + F_{A_0}) \right) \\ &= \frac{1}{2} \int_M \left(\langle \gamma^{-1}\psi, \gamma^{-1}\mathcal{D}_A\psi \rangle dv_g + (A - A_0) \wedge (F_A + F_{A_0}) \right) \\ &\quad + \int_M \gamma^{-1}d\gamma \wedge (F_A + F_{A_0}) \end{aligned}$$

Since γ^{-1} acts unitary and $[F_A] = [F_{A_0}] = 2\pi ic(\sigma)$,

$$\text{csd}(\gamma \cdot (\psi, A)) = \text{csd}(\psi, A) - 8\pi^2 \int_M \left[\frac{1}{2\pi i} \gamma^{-1} d\gamma \right] \wedge c(\sigma). \quad \square$$

According to (D, 15), the class $[\frac{1}{2\pi i} \gamma^{-1} d\gamma]$ belongs to $H_{dR}^1(M; \mathbb{Z})$ which is the image of $H^1(M; \mathbb{Z})$ in $H_{dR}^1(M; \mathbb{R})$. The same applies to $c(\sigma)$. Therefore, the integral over $[\frac{1}{2\pi i} \gamma^{-1} d\gamma] \wedge c(\sigma)$ is integer valued and vanishes for all γ only if $c(\sigma)$ is a torsion class.

Hence in the general case, the Chern-Simons-Dirac functional descends to \mathcal{B} as a function with values in $\mathbb{R}/(8\pi^2\mathbb{Z})$. Whenever we want to refer to this phenomenon, we shall usually view csd as a function

$$\text{csd} : \mathcal{B} \rightarrow S^1$$

We now want to linearize the action of the group of gauge transformations.⁶ As the Lie algebra of U_1 is $i\mathbb{R}$, infinitesimal gauge transformations are smooth maps $M \rightarrow i\mathbb{R}$.

Proposition (I, 2.8). *If $(\psi, A) \in \mathcal{C}$, the “derivative” of the action map*

$$\mathcal{G} \rightarrow \mathcal{C}, \quad \gamma \mapsto \gamma \cdot (\psi, A)$$

at $\gamma = 1$ is the first-order differential operator

$$G_{(\psi, A)} : C^\infty(M, i\mathbb{R}) \rightarrow C^\infty(M, E),$$

given by

$$G_{(\psi, A)}(f) := \left. \frac{d}{dt} \right|_{t=0} \exp(tf) \cdot (\psi, A) = (-f\psi, 2df). \quad (\text{I, } 10)$$

The formal adjoint of $G_{(\psi, A)}$ with respect to the L^2 scalar products is

$$G_{(\psi, A)}^*(\varphi, a) = 2d^*a - i \operatorname{Im}\langle \varphi, \psi \rangle. \quad (\text{I, } 11)$$

Proof. Let $f \in C^\infty(M, i\mathbb{R})$. Then

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \exp(tf) \cdot (\psi, A) &= \left. \frac{d}{dt} \right|_{t=0} (\exp(-tf)\psi, A + 2\exp(-tf)d\exp(tf)) \\ &= (-f\psi, 2df). \end{aligned}$$

To calculate the formal adjoint of $G_{(\psi, A)}$, we now let $(\varphi, a) \in C^\infty(M, E)$. Then, recalling that f is imaginary valued, we find

$$\begin{aligned} ((\varphi, a), G_{(\psi, A)}(f))_{L^2} &= \operatorname{Re}(\varphi, -f\psi)_{L^2} + (a, 2df)_{L^2} \\ &= (2d^*a - i \operatorname{Im}\langle \varphi, \psi \rangle, f)_{L^2}. \quad \square \end{aligned}$$

⁶In the next chapter we shall see that if we consider suitable Sobolev completions, the group of gauge transformations is a Banach Lie group which acts smoothly on the configuration space. Hence, taking the differential of the action is meaningful. For the time being, we will perform the involved calculations only formally.

The elliptic complex. If (ψ, A) is a monopole, then $\text{SW}(\gamma \cdot (\psi, A)) = 0$ for every $\gamma \in \mathcal{G}$. Taking derivatives at $\gamma = 1$ yields $F_{(\psi, A)} \circ G_{(\psi, A)} = 0$, and we obtain a complex

$$0 \rightarrow C^\infty(M, i\mathbb{R}) \xrightarrow{G} C^\infty(M, E \oplus i\mathbb{R}) \xrightarrow{F+G} C^\infty(M, E) \rightarrow 0, \quad (\text{I, 12})$$

where we denote the map $f \mapsto (G(f), 0)$ a little inaccurately also by G . This is a complex of first-order differential operators. Associated to (I, 12) there is the rolled-up operator

$$T_{(\psi, A)} := (F_{(\psi, A)} \oplus G_{(\psi, A)}, G_{(\psi, A)}^*) : C^\infty(M, E \oplus i\mathbb{R}) \rightarrow C^\infty(M, E \oplus i\mathbb{R}). \quad (\text{I, 13})$$

This operator is well-defined, irrespective of whether (ψ, A) is a SW-monopole or not. Explicitly, it is given by

$$T_{(\psi, A)} \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} = \begin{pmatrix} \mathcal{D}_A & 0 & 0 \\ 0 & -*d & 2d \\ 0 & 2d^* & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} + \begin{pmatrix} \frac{1}{2}c(a)\psi - f\psi \\ q(\psi, \varphi) \\ -i \text{Im}\langle \varphi, \psi \rangle \end{pmatrix}. \quad (\text{I, 14})$$

Proposition (I, 2.9). *For each configuration (ψ, A) , the differential operator $T_{(\psi, A)}$ is elliptic and formally self-adjoint. Hence, if (ψ, A) is a monopole, the complex (I, 12) is elliptic, i.e., the associated sequence of principal symbols is exact.*

Proof. To prove ellipticity, we only have to consider the first-order term of $T_{(\psi, A)}$. According to the explicit description (I, 14), this term splits into the elliptic operator $\mathcal{D}_A : C^\infty(M, S) \rightarrow C^\infty(M, S)$ and the operator

$$\begin{pmatrix} -*d & 2d \\ 2d^* & 0 \end{pmatrix} : C^\infty(M, iT^*M \oplus i\mathbb{R}) \rightarrow C^\infty(M, iT^*M \oplus i\mathbb{R}). \quad (\text{I, 15})$$

To prove ellipticity of the latter operator we use the rolled-up operator of the deRham complex

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \rightarrow 0.$$

This is an elliptic operator given by

$$\begin{pmatrix} d^* & d \\ d & 0 \end{pmatrix} = \begin{pmatrix} *d^* & d \\ d & 0 \end{pmatrix} : \Omega^2 \oplus \Omega^0 \rightarrow \Omega^1 \oplus \Omega^3.$$

Here, we are using the explicit description of d^* on 3-manifolds. The operator (I, 15) is elliptic because it can be obtained from the above elliptic operator by a combination with bundle isomorphisms in the following way:

$$\Omega^1 \oplus \Omega^0 \xrightarrow{\begin{pmatrix} * & 0 \\ 0 & -2\text{id} \end{pmatrix}} \Omega^2 \oplus \Omega^0 \xrightarrow{\begin{pmatrix} *d^* & d \\ d & 0 \end{pmatrix}} \Omega^1 \oplus \Omega^3 \xrightarrow{\begin{pmatrix} -\text{id} & 0 \\ 0 & -2^* \end{pmatrix}} \Omega^1 \oplus \Omega^0.$$

Therefore, the first-order term of $T_{(\psi,A)}$ is the direct sum of two elliptic operators, which shows that $T_{(\psi,A)}$ is also elliptic. The assertion about formal self-adjointness is an immediate consequence of formal self-adjointness of $F_{(\psi,A)}$. \square

Remark. In four dimensional Seiberg-Witten theory, the geometric origin of the complex corresponding to (I, 12) is more transparent since one does not have to add the term G to the operator F in order to have ellipticity. Later, when we are going to study the local structure of the moduli space in Section II.4, the nature of (I, 12) will become clearer.

Chapter II

The Structure of the Moduli Space

To understand the structure of the Seiberg-Witten moduli space, we have to endow the configuration space and the group of gauge transformations with suitable topologies. As the objects of our study are solutions of a system of partial differential equations, it is natural to do this via Sobolev spaces. We can then exploit the powerful machinery provided by the theory of elliptic equations on compact manifolds to prove remarkable topological properties. The material we need is summarized in Appendix A.

The organization of this chapter is as follows. Section 1 introduces the functional analytic setting. It turns out that \mathcal{G} can be made a Banach Lie group acting smoothly on \mathcal{C} . In Section 2, we analyse the action of \mathcal{G} on \mathcal{C} using the situation occurring for proper actions of finite dimensional Lie groups as a guideline. We establish the Hausdorff property of the quotient and a slice theorem which shall provide the irreducible part of \mathcal{B} with the structure of a Banach manifold.

Moreover, we will see in Section 3 that the moduli space \mathcal{M} is a sequentially compact subset of \mathcal{B} . This contrasts the corresponding result in instanton theory, where the moduli space has to be compactified through a complicated procedure (cf. Donaldson & Kronheimer [15] or Freed & Uhlenbeck [17]). This is one of the reasons why Seiberg-Witten theory is considered as a simplification of Donaldson theory. Making use of the implicit function theorem we will then observe in Section 4 that the irreducible part of the moduli space is usually expected to be a submanifold of dimension zero. Therefore, in the absence of reducible monopoles, \mathcal{M} consists solely of finitely many points. Finally, we shall use this observation to define an orientation of the moduli space in Section 5. There is a general procedure, introduced by Donaldson in [12], to endow gauge theoretical moduli spaces

with an orientation. This applies to Seiberg-Witten theory as well. Using the Chern-Simons-Dirac functional, we shall then interpret the signed count of monopoles obtained in this way as an Euler characteristic associated to the irreducible part of the quotient \mathcal{B} .

The presentation of the topological aspects we are giving is an imitation of the corresponding results in four dimensional Seiberg-Witten theory as they are presented, for example, in the monographs by J. Morgan [42] and L.I. Nicolaescu [45]. The discussion of the local structure and the orientation of the moduli space follows the work of C.H. Taubes [52] and W. Chen [11].

1 Functional analytic set-up

Suppose that M is a closed, oriented Riemannian 3-manifold with spin^c structure σ . Let $L_1^2(M, S)$ denote the sections of the spinor bundle $S(\sigma)$ which are of Sobolev class 1, and let $\mathcal{A}_1(\sigma)$ be the affine Hilbert space of L_1^2 -connections on $L(\sigma)$, i.e.,

$$\mathcal{A}_1(\sigma) := \{A_0 + a \mid a \in L_1^2(M, iT^*M)\},$$

where A_0 is a fixed C^∞ gauge field. Because of the affine structure of \mathcal{A} , the definition of $\mathcal{A}_1(\sigma)$ is independent of the particular choice of A_0 . Then the configuration space

$$\mathcal{C}_1(\sigma) := L_1^2(M, S(\sigma)) \times \mathcal{A}_1(\sigma)$$

is a real affine Hilbert space modelled on $L_1^2(M, S \oplus iT^*M)$. As the group of gauge transformations we now take

$$\mathcal{G}_2 := L_2^2(M, U_1),$$

which is the set of functions $\gamma : M \rightarrow \mathbb{C}$ of Sobolev class 2 that take values in U_1 . Note that this definition makes sense since on 3-manifolds, L_2^2 embeds in C^0 (cf. Theorem (A, 1.1)). The moduli space \mathcal{M} carries the topology induced by the quotient topology on $\mathcal{B}_1 = \mathcal{C}_1/\mathcal{G}_2$.

Remark. As we shall see below, the Sobolev orders we are choosing are motivated by the consideration in Example (A, 1.4): It must be guaranteed that there are continuous Sobolev multiplications

$$L_k^2 \times L_k^2 \rightarrow L_k^2 \quad \text{and} \quad L_k^2 \times L_l^2 \rightarrow L_l^2,$$

where k and l are the Sobolev orders associated to gauge transformations and configurations respectively. Since this depends on the dimension of the

underlying manifold, one has to choose k and l differently in four dimensional Seiberg-Witten theory. However, some authors assume higher Sobolev orders in the three dimensional case as well which simplifies some proofs. Although we will see in Section 3 that the structure of the moduli space does not depend on the particular choice, we do not avoid the slightly bigger effort of working with the lowest possible Sobolev orders as this will allow some of the involved differential operators to be defined on their natural domains.

Differentiability properties. We now want to establish the basic set-up for performing calculus in the given framework.

Lemma (II, 1.1). *The quadratic map $\psi \mapsto q(\psi)$ induces a smooth map*

$$q : L_1^2(M, S) \rightarrow L^2(M, iT^*M).$$

*If $k \geq 2$, we obtain a smooth map $q : L_k^2(M, S) \rightarrow L_k^2(M, iT^*M)$.*

Proof. Since M is three dimensional, Proposition (A, 1.3) guarantees that there is a bounded Sobolev multiplication $L_1^2 \times L_1^2 \rightarrow L^2$. The first assertion then immediately follows because $q(\psi)$ is a quadratic expression in ψ . For $k \geq 2$ we deduce from Example (A, 1.4) that there is a bounded Sobolev multiplication $L_k^2 \times L_k^2 \rightarrow L_k^2$ associated to any bilinear map. This yields smoothness of $q : L_k^2(M, S) \rightarrow L_k^2(M, iT^*M)$ in this case as well. \square

Proposition (II, 1.2). *The Chern-Simons-Dirac functional $\text{csd} : \mathcal{C}_1 \rightarrow \mathbb{R}$ is a smooth map. Its L^2 -gradient SW extends naturally to a smooth map $\mathcal{C}_1 \rightarrow L^2(M, E)$. For any $(\psi, A) \in \mathcal{C}_1$, the Hessian $F_{(\psi, A)}$ defines a symmetric operator in $L^2(M, E)$ with domain $L_1^2(M, E)$.*

Proof. Since $\int_M : L^1(M, \Lambda^3 T^*M) \rightarrow \mathbb{R}$ is smooth, we have to establish that the integrand of the Chern-Simons-Dirac functional is a smooth map $\mathcal{C}_1 \rightarrow L^1(M, \Lambda^3 T^*M)$. For this, we have to prove first that

$$\mathcal{C}_1 \rightarrow L^1(M, \mathbb{R}), (\psi, A) \mapsto \text{Re}\langle \psi, \mathcal{D}_A \psi \rangle$$

is smooth. Smoothness of the multiplication $L^2 \times L^2 \rightarrow L^1$ shows that it suffices to establish that for a fixed C^∞ gauge field A_0 , the map

$$L_1^2(M, E) \rightarrow L^2(M, S), (\psi, a) \mapsto \mathcal{D}_{A_0} \psi + \frac{1}{2}c(a)\psi.$$

is smooth. \mathcal{D}_{A_0} induces a bounded linear—and consequently a smooth—map $L_1^2(M, S) \rightarrow L^2(M, S)$. As we have already seen before, Proposition (A, 1.3)

shows that the second term also yields a smooth map $L_1^2(M, E) \rightarrow L^2(M, E)$. Secondly, we have to show that

$$\mathcal{A}_1 \rightarrow L^1(M, \Lambda^3 T^* M), \quad A \mapsto (A - A_0) \wedge (F_A + F_{A_0}),$$

is smooth. Since there is a Sobolev multiplication $L_1^2 \times L^2 \rightarrow L^1$, this easily follows from smoothness of

$$L_1^2(M, iT^* M) \rightarrow L^2(M, \Lambda^2 iT^* M), \quad a \mapsto F_{A_0} + da.$$

From the above considerations and Lemma (II, 1.1), one also concludes that the L^2 gradient SW extends to a smooth map $\mathcal{C}_1 \rightarrow L^2(M, E)$. Since the computations of Proposition (I, 2.7) in the last chapter remain valid for L_1^2 configurations, the extension of SW is the L^2 gradient of $\text{csd} : \mathcal{C}_1 \rightarrow \mathbb{R}$.¹ Moreover, the differential of SW at $(\psi, A) \in \mathcal{C}_1$ is again given by formula (I, 8). Since this is a formally self-adjoint first-order differential operator, it extends to an unbounded symmetric operator in $L^2(M, E)$ with natural domain $L_1^2(M, E)$. Notice that the zero-order term of $F_{(\psi, A)}$ is possibly non-smooth. Then, however, it is easy to check that the multiplication rule $L_1^2 \times L_1^2 \rightarrow L^2$ guarantees that $F_{(\psi, A)}$ is still well-defined as a bounded operator $L_1^2(M, E) \rightarrow L^2(M, E)$. \square

Proposition (II, 1.3). *The group of gauge transformations \mathcal{G}_2 is a Banach Lie group modelled on $L_2^2(M, i\mathbb{R})$, and its action on \mathcal{C}_1 is smooth.*

Proof. Sobolev multiplication (A, 1.4) guarantees that multiplication of complex functions on M extends to a smooth bilinear map

$$L_2^2(M, \mathbb{C}) \times L_2^2(M, \mathbb{C}) \rightarrow L_2^2(M, \mathbb{C}).$$

Like in finite dimensional Lie group theory, the implicit function theorem—which is also valid in Banach spaces—guarantees that taking the inverse of an invertible function $f \in L_2^2(M, \mathbb{C})$ is a smooth map, defined on the subset

$$L_2^2(M, \mathbb{C}^*) := \{f \in L_2^2(M, \mathbb{C}) \mid \forall_{x \in M} : f(x) \in \mathbb{C}^*\}.$$

Note that this set is well-defined since there is a continuous embedding of $L_2^2(M, \mathbb{C})$ in $C^0(M, \mathbb{C})$. Moreover, with respect to this embedding, $L_2^2(M, \mathbb{C}^*)$ is the preimage of the open subset $C^0(M, \mathbb{C}^*)$ of $C^0(M, \mathbb{C})$ and is therefore open in $L_2^2(M, \mathbb{C})$. Hence, $L_2^2(M, \mathbb{C}^*)$ is a Banach Lie group.

¹However, SW is not a gradient vector field with respect to the natural metric on \mathcal{C}_1 , given by the L_1^2 scalar product on $L_1^2(M, E)$.

We now want to show that it contains $\mathcal{G}_2 = L_2^2(M, U_1)$ as a closed Lie subgroup. For this it suffices to establish that \mathcal{G}_2 is a closed submanifold of $L_2^2(M, \mathbb{C}^*)$. This shall be done by constructing local charts of $L_2^2(M, \mathbb{C}^*)$ mapping \mathcal{G}_2 to the real subspace $L_2^2(M, i\mathbb{R})$ of $L_2^2(M, \mathbb{C})$. Note that this subspace is closed—again because of the embedding of L_2^2 in C^0 .

Let $\mathbb{C}^- := \mathbb{C} \setminus (-\infty, 0]$. Then the set

$$L_2^2(M, \mathbb{C}^-) := \{f \in L_2^2(M, \mathbb{C}) \mid \forall_{x \in M} : f(x) \in \mathbb{C}^-\}$$

is an open subset of $L_2^2(M, \mathbb{C}^*)$. If we let

$$V := \mathbb{R} \times (-i\pi, i\pi) \subset \mathbb{C},$$

then $\exp|_V : V \rightarrow \mathbb{C}^-$ is a diffeomorphism. This in turn induces a diffeomorphism

$$\exp : L_2^2(M, V) \rightarrow L_2^2(M, \mathbb{C}^-), \quad f \mapsto \exp(f),$$

where $L_2^2(M, V)$ is defined in the same manner as $L_2^2(M, \mathbb{C}^*)$ and $L_2^2(M, \mathbb{C}^-)$ above. Clearly,

$$\exp \left(L_2^2(M, V) \cap L_2^2(M, i\mathbb{R}) \right) = L_2^2(M, \mathbb{C}^-) \cap L_2^2(M, U_1).$$

Multiplication by an element of $L_2^2(M, U_1)$ induces a diffeomorphism of $L_2^2(M, \mathbb{C}^*)$. Hence, for arbitrary $\gamma \in L_2^2(M, U_1)$, the set $\gamma \cdot L_2^2(M, \mathbb{C}^-)$ is an open neighbourhood of γ in $L_2^2(M, \mathbb{C}^*)$. As a consequence,

$$\gamma \cdot \exp : L_2^2(M, V) \rightarrow \gamma \cdot L_2^2(M, \mathbb{C}^-), \quad f \mapsto \gamma \cdot \exp(f),$$

is a diffeomorphism satisfying

$$(\gamma \cdot \exp)(L_2^2(M, V) \cap L_2^2(M, i\mathbb{R})) = (\gamma \cdot L_2^2(M, \mathbb{C}^-)) \cap L_2^2(M, U_1).$$

Since $L_2^2(M, U_1)$ can be covered by sets of the above type, it is a closed submanifold of $L_2^2(M, \mathbb{C}^*)$ modelled on $L_2^2(M, i\mathbb{R})$.

According to Example (A, 1.4), there is a smooth multiplication

$$L_2^2 \times L_1^2 \rightarrow L_1^2.$$

Hence, the action of \mathcal{G}_2 on $L_1^2(M, S)$, which is given by $(\gamma, \psi) \mapsto \gamma^{-1}\psi$, is smooth. \mathcal{G}_2 acts on the space of gauge fields via $(\gamma, A) \mapsto A + 2\gamma^{-1}d\gamma$. As $d : L_2^2(M, \mathbb{C}) \rightarrow L_1^2(M, \mathbb{C})$ is a bounded linear operator, it follows again from Example (A, 1.4) that the map

$$L_2^2(M, U_1) \rightarrow L_1^2(M, iT^*M), \quad \gamma \mapsto \gamma^{-1}d\gamma,$$

is smooth. Thus, the Seiberg-Witten configuration space is acted on smoothly by \mathcal{G}_2 . \square

We are now in the position to make the considerations in Proposition (I, 2.8) more precise. Let $(\psi, A) \in \mathcal{C}_1$. Since the action of \mathcal{G}_2 on \mathcal{C}_1 is smooth, we may take the derivative of the map

$$\mathcal{G}_2 \rightarrow \mathcal{C}_1, \quad \gamma \mapsto \gamma \cdot (\psi, A).$$

The formal calculation in loc. cit. shows that this results in the bounded linear map

$$G_{(\psi, A)} : L_2^2(M, i\mathbb{R}) \rightarrow L_1^2(M, E), \quad f \mapsto (-f\psi, 2df),$$

where—as always—we are using the abbreviation

$$E := S \oplus iT^*M.$$

Notice that $G_{(\psi, A)}$ above is not defined on its natural domain $L_1^2(M, i\mathbb{R})$. We shall, however, consider $G_{(\psi, A)}$ as a closed operator $L^2(M, i\mathbb{R}) \rightarrow L^2(M, E)$ with domain $L_1^2(M, i\mathbb{R})$ restricting to $L_2^2(M, i\mathbb{R})$ whenever it is necessary. In the same way, $G_{(\psi, A)}^*$ shall always denote the *functional analytic* adjoint of $G_{(\psi, A)}$. When restricted to $L_1^2(M, E)$ it coincides with the natural extension of the formal adjoint (I, 11).

We will now turn to the extension of the elliptic operator $T_{(\psi, A)}$ which was defined in (I, 13) for smooth (ψ, A) as

$$T_{(\psi, A)} = (F_{(\psi, A)} + G_{(\psi, A)}, G_{(\psi, A)}^*) : C^\infty(M, E \oplus i\mathbb{R}) \rightarrow C^\infty(M, E \oplus i\mathbb{R}).$$

Then $T_{(\psi, A)}$ is also well-defined as an operator in $L^2(M, E \oplus i\mathbb{R})$ with natural domain $L_1^2(M, E \oplus i\mathbb{R})$. Our aim is to show that $T_{(\psi, A)}$ is a self-adjoint operator with compact resolvent. Fixing a smooth gauge field A_0 and writing $A = A_0 + a_0$, we let

$$K_{(\psi, a_0)} := T_{(\psi, A_0 + a_0)} - T_{(0, A_0)}.$$

Since $\mathcal{D}_A = \mathcal{D}_{A_0} + \frac{1}{2}c(a_0)$, formula (I, 14) shows that

$$K_{(\psi, a_0)} \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c(a_0)\varphi + \frac{1}{2}c(a)\psi - f\psi \\ q(\psi, \varphi) \\ -i \operatorname{Im}\langle \varphi, \psi \rangle \end{pmatrix}, \quad \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} \in L_1^2(M, E \oplus i\mathbb{R}).$$

Using the considerations at the end of Appendix A as a guideline, we now need to ascertain the following:

Lemma (II, 1.4). *For any $(\psi, a_0) \in L_1^2(M, E)$, the operator*

$$K_{(\psi, a_0)} : L_1^2(M, E \oplus i\mathbb{R}) \rightarrow L^2(M, E \oplus i\mathbb{R})$$

is compact and symmetric with respect to the L^2 scalar product.

Proof. From the explicit description of $K_{(\psi, a_0)}$ we deduce that the assertion of the lemma reduces to the claim that multiplication by an element of L_1^2 yields a compact operator $L_1^2 \rightarrow L^2$.

Carefully checking the assumptions of Proposition (A, 1.3), we obtain a continuous Sobolev multiplication

$$L_1^2 \times L_1^2 \rightarrow L_1^p,$$

provided that $p \in (1, \frac{3}{2})$. If in addition $p > \frac{6}{5}$, we have $1 - \frac{3}{p} > -\frac{3}{2}$, and the Rellich-Kondrachov Theorem (A, 1.2) implies that L_1^p embeds compactly in L^2 . Then the desired compactness property follows. The symmetry with respect to the L^2 scalar product is an immediate consequence of the definition of $K_{(\psi, a_0)}$ as the difference of two symmetric operators. \square

Invoking Theorem (A, 2.9) about relative compact perturbations of operators with compact resolvent, we can now draw an important conclusion:

Proposition (II, 1.5). *Let $(\psi, A) \in \mathcal{C}_1$. Then the operator $T_{(\psi, A)}$ induces a self-adjoint operator in $L^2(M, E \oplus i\mathbb{R})$ with domain $L_1^2(M, E \oplus i\mathbb{R})$, i.e., using the notation of (C, 3),*

$$T_{(\psi, A)} \in \mathcal{L}_{sa}(L_1^2(M, E \oplus i\mathbb{R}), L^2(M, E \oplus i\mathbb{R})).$$

Moreover, $T_{(\psi, A)}$ has compact resolvent and thus discrete spectrum. In particular, it is a Fredholm operator.

According to the considerations in App. C, Sec. 1, \mathcal{L}_{sa} is an open subset of the Banach space \mathcal{L}_{sym} which is the space of symmetric operators with the same fixed domain. Hence \mathcal{L}_{sa} inherits the structure of a Banach manifold if endowed with the operator norm topology. With respect to this, we have:

Proposition (II, 1.6). *The assignment $(\psi, A) \mapsto T_{(\psi, A)}$ defines a smooth map*

$$\mathcal{C}_1(\sigma) \rightarrow \mathcal{L}_{sa}(L_1^2(M, E \oplus i\mathbb{R}), L^2(M, E \oplus i\mathbb{R})),$$

Proof. Since \mathcal{L}_{sa} is an open subset of \mathcal{L}_{sym} , it suffices to insure that the assignment

$$L_1^2(M, E) \rightarrow \mathcal{L}_{sym}(L_1^2(M, E \oplus i\mathbb{R}), L^2(M, E \oplus i\mathbb{R})), \quad (\psi, a_0) \mapsto K_{(\psi, a_0)}$$

is smooth. By linearity of this map, smoothness is equivalent to continuity.

Using the continuous Sobolev multiplication $L_1^2 \times L_1^2 \rightarrow L^2$, one straightforwardly obtains

$$\|K_{(\psi, a_0)}(\varphi, a, f)\|_{L^2} \leq \text{const} \cdot \|(\psi, a_0)\|_{L_1^2} \cdot \|(\varphi, a, f)\|_{L_1^2}.$$

Therefore, we can estimate the operator norm by

$$\|K_{(\psi, a_0)}\|_{L_1^2, L^2} = \sup_{(\varphi, a, f)} \frac{\|K_{(\psi, a_0)}(\varphi, a, f)\|_{L^2}}{\|(\varphi, a, f)\|_{L_1^2}} \leq \text{const} \cdot \|(\psi, a_0)\|_{L_1^2}$$

which ensures continuity of $(\psi, a_0) \mapsto K_{(\psi, a_0)}$. \square

2 Topology of the quotient space

Our next aim is to investigate the topology of the set of configurations modulo gauge equivalence, i.e., of the quotient $\mathcal{B}_1 = \mathcal{C}_1/\mathcal{G}_2$. The first observation is that \mathcal{B}_1 is second countable since its topology is defined as the quotient of an affine space modelled on a separable Hilbert space.

The Hausdorff property. When studying the quotient of group actions $G \times X \rightarrow X$, the situation simplifies if G acts *properly* on X , i.e., if the map

$$G \times X \rightarrow X \times X, \quad (g, x) \mapsto (x, gx)$$

is proper.² Recall (e.g. from [55], Sec. I.3) that the quotient X/G of a proper group action is always a Hausdorff space. The following (simple) criterium is useful in our context:

Lemma (II, 2.1). *Let $G \times X \rightarrow X$ be a (topological) group action. Suppose that all stabilizers are compact and that for all sequences (x_n) in X and (g_n) in G the following holds:*

If $x_n \rightarrow x$ and $g_n x_n \rightarrow y$, then there exists a convergent subsequence of (g_n) whose limit point $g \in G$ satisfies $gx = y$.

Then G acts properly on X .

Proposition (II, 2.2). *The group \mathcal{G}_2 acts properly on \mathcal{C}_1 . In particular, the quotient \mathcal{B}_1 is a Hausdorff space.*

Proof. We use the criterium in Lemma (II, 2.1). Let $((\psi_n, A_n))$ and (γ_n) be sequences in \mathcal{C}_1 and \mathcal{G}_2 respectively. Suppose that there exists (ψ_0, A_0) and (ψ, A) in \mathcal{C}_1 such that

$$(\psi_n, A_n) \xrightarrow{n \rightarrow \infty} (\psi_0, A_0) \quad \text{and} \quad \gamma_n \cdot (\psi_n, A_n) \xrightarrow{n \rightarrow \infty} (\psi, A).$$

²This means that the map is closed and that preimages of points are compact

In particular,

$$\|\gamma_n^{-1}\psi_n - \psi\|_{L_1^2} \xrightarrow{n \rightarrow \infty} 0 \quad (\text{II, 1})$$

and, if we let $a_n := A_n - A$,

$$\|A_n + 2\gamma_n^{-1}d\gamma_n - A\|_{L_1^2} = \|a_n + 2\gamma_n^{-1}d\gamma_n\|_{L_1^2} \xrightarrow{n \rightarrow \infty} 0. \quad (\text{II, 2})$$

Since multiplication with a gauge transformation does not change the value of $\|\cdot\|_{L^p}$, we deduce from the continuous embedding $L_1^2 \hookrightarrow L^4$ that

$$\begin{aligned} \|d\gamma_n\|_{L^4} &= \|\gamma_n^{-1}d\gamma_n\|_{L^4} \leq \text{const} \cdot \|\gamma_n^{-1}d\gamma_n\|_{L_1^2} \\ &\leq \text{const} \cdot (\|a_n\|_{L_1^2} + \|a_n + 2\gamma_n^{-1}d\gamma_n\|_{L_1^2}), \end{aligned} \quad (\text{II, 3})$$

where in the last line we have employed the triangle inequality. As a convergent sequence, (a_n) is bounded in L_1^2 . We thus conclude from (II, 2) that $(d\gamma_n)$ is a bounded sequence in L^4 (and hence also in L^2). To obtain an L_2^2 -bound on (γ_n) it remains to consider the sequence of the second derivatives. Viewing (γ_n) as a sequence in $L_2^2(M; \mathbb{C})$, we have

$$\nabla^2 \gamma_n = \nabla(d\gamma_n) \in L^2(M, T^*M^{\otimes 2} \otimes \mathbb{C}).$$

Note that

$$\nabla(d\gamma_n) = \gamma_n \nabla(\gamma_n^{-1}d\gamma_n) - \gamma_n d\gamma_n^{-1} \otimes d\gamma_n$$

Since $d\gamma_n^{-1} = -\gamma_n^{-2}d\gamma_n$, the (pointwise) norm of the second summand is equal to $|d\gamma_n|^2$. Therefore,

$$\begin{aligned} \|\nabla(d\gamma_n)\|_{L^2} &\leq \|\nabla(\gamma_n^{-1}d\gamma_n)\|_{L^2} + \|d\gamma_n^{-1} \otimes d\gamma_n\|_{L^2} \\ &\leq \|\gamma_n^{-1}d\gamma_n\|_{L_1^2} + \|d\gamma_n\|_{L^4}^2. \end{aligned} \quad (\text{II, 4})$$

As we have seen in (II, 3), the right hand side of (II, 4) is bounded so that we obtain the desired L_2^2 -bound on (γ_n) .

The Rellich-Kondrachov Theorem (A, 1.2) now implies that L_2^2 embeds compactly in L_1^2 . We can thus find a subsequence of (γ_n) which converges in L_1^2 . Additionally, according to (II, 4), the sequence $(\nabla(d\gamma_n))$ is bounded in the Hilbert space $L^2(M, T^*M^{\otimes 2} \otimes \mathbb{C})$. Hence, (γ_n) contains a subsequence such that the second derivatives are weakly convergent in L^2 . Without loss of generality, we may therefore assume that (γ_n) converges strongly in L_1^2 to, say, γ and that $(\nabla(d\gamma_n))$ converges weakly in L^2 to, say, η . This implies that $\nabla(d\gamma) = \eta$ weakly in L^2 . As $\nabla \circ d$ is injectively elliptic, we have $\gamma \in L_2^2(M, U_1)$.

It remains to show that $\gamma \cdot (\psi_0, A_0) = (\psi, A)$, i.e., with $a := A_0 - A$, that

$$\gamma^{-1}\psi_0 = \psi \quad \text{and} \quad 2d\gamma + \gamma a = 0.$$

Equations (II, 1) and (II, 2) guarantee that $(\gamma_n^{-1}\psi_n) \rightarrow \psi$ in L^2 as well as $2d\gamma_n + \gamma_n a_n \rightarrow 0$ in L^2 . On the other hand, continuity of the multiplication $L_1^2 \times L_1^2 \rightarrow L^2$ shows that the sequence $(\gamma_n^{-1}\psi_n)$ converges to $\gamma^{-1}\psi_0$ in L^2 , and $d\gamma_n \rightarrow d\gamma$ as well as $\gamma_n a_n \rightarrow \gamma a$ in L^2 . Hence, uniqueness of the limit points implies the above formulæ. \square

Local slices for the action. In this paragraph we shall establish a slice theorem for the action of \mathcal{G}_2 on \mathcal{C}_1 , analogous to the well-known situation from the theory of finite dimensional Lie group actions (cf. [55], Sec. I.5): For every $(\psi, A) \in \mathcal{C}_1$ we are looking for a subspace

$$S_{(\psi, A)} \subset T_{(\psi, A)}\mathcal{C}_1 = L_1^2(M, E),$$

complementary to the tangent space of the orbit $\mathcal{G}_2 \cdot (\psi, A)$. We wish to model nearby orbits by $\mathcal{G}_2 \times S_{(\psi, A)}$, making use of the natural map

$$\begin{aligned} \pi : \mathcal{G}_2 \times S_{(\psi, A)} &\rightarrow \mathcal{C}_1, & \pi(\gamma, (\varphi, a)) &= \gamma \cdot ((\psi, A) + (\varphi, a)) \\ & & &= (\gamma^{-1}\psi + \gamma^{-1}\varphi, A + a + 2\gamma^{-1}d\gamma). \end{aligned} \quad (\text{II, 5})$$

Clearly, if (ψ, A) is a reducible configuration, the map π cannot be injective. Therefore, $S_{(\psi, A)}$ has to be chosen to be invariant under the natural action of the stabilizer $\mathcal{G}_{(\psi, A)}$. We will then have to study

$$\mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} S_{(\psi, A)} := (\mathcal{G}_2 \times S_{(\psi, A)}) / \mathcal{G}_{(\psi, A)}.$$

Since the action of the stabilizer on $T_{(\psi, A)}\mathcal{C}_1 = L_1^2(M, E)$ is orthogonal with respect to the L^2 metric, a natural choice for $S_{(\psi, A)}$ is provided by taking the orthogonal complement of the tangent space to the gauge orbit. This tangent space is essentially the image of the differential of action map, i.e., the image of (cf. (I, 10))

$$G_{(\psi, A)} : L_2^2(M, i\mathbb{R}) \rightarrow L_1^2(M, E), \quad f \mapsto (-f\psi, 2df).$$

As the leading term of $G_{(\psi, A)}$ is injectively elliptic, we infer from the Hodge decomposition (A, 5) that there is an L^2 -orthogonal splitting

$$L_1^2(M, E) = \text{im}(G_{(\psi, A)}|_{L_2^2}) \oplus \ker(G_{(\psi, A)}^*|_{L_1^2}). \quad (\text{II, 6})$$

Recall from Proposition (I, 2.8) that for $(\varphi, a) \in L_1^2(M, E)$,

$$G_{(\psi, A)}^*(\varphi, a) = 2d^*a - i \text{Im}\langle \varphi, \psi \rangle.$$

Definition (II, 2.3). For all $(\psi, A) \in \mathcal{C}_1$ we define the *local slice* of the \mathcal{G}_2 -action at the point (ψ, A) as the subspace

$$S_{(\psi, A)} := \ker(G_{(\psi, A)}^*|_{L_1^2}) \subset L_1^2(M, E).$$

Lemma (II, 2.4). *For every $(\psi, A) \in \mathcal{C}_1$, the local slice $S_{(\psi, A)}$ is $\mathcal{G}_{(\psi, A)}$ -invariant.*

Proof. If (ψ, A) is irreducible, the stabilizer is trivial. For reducible (ψ, A) , i.e., if $\psi = 0$, we have

$$S_{(\psi, A)} = L_1^2(M, S) \oplus \ker(d^*|_{L_1^2}). \quad (\text{II, 7})$$

Recall from (I, 6) that \mathcal{G}_2 , and hence also $\mathcal{G}_{(\psi, A)}$, acts only on the spinor part of $L_1^2(M, E)$. Therefore, (II, 7) is invariant under the action of $\mathcal{G}_{(\psi, A)}$. \square

The stabilizer acts on $\mathcal{G}_2 \times S_{(\psi, A)}$ via

$$\lambda \cdot (\gamma, (\varphi, a)) := (\gamma\lambda^{-1}, \lambda \cdot (\varphi, a)) = (\gamma\lambda^{-1}, (\lambda^{-1}\varphi, a)), \quad \lambda \in \mathcal{G}_{(\psi, A)}.$$

One readily checks that the natural map (II, 5) is $\mathcal{G}_{(\psi, A)}$ -invariant thus factoring to a map

$$\bar{\pi} : \mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} S_{(\psi, A)} \longrightarrow \mathcal{C}_1.$$

To lift the \mathcal{G}_2 -action on \mathcal{C}_1 to $\mathcal{G}_2 \times S_{(\psi, A)}$ we let \mathcal{G}_2 act from the left on the first factor, i.e., for $(\gamma, (\varphi, a)) \in \mathcal{G}_2 \times S_{(\psi, A)}$ and $\gamma' \in \mathcal{G}_2$ we let

$$\gamma' \cdot (\gamma, (\varphi, a)) := (\gamma'\gamma, (\varphi, a)).$$

Then π is clearly \mathcal{G}_2 -equivariant. Moreover, the actions of \mathcal{G}_2 and $\mathcal{G}_{(\psi, A)}$ on $\mathcal{G}_2 \times S_{(\psi, A)}$ commute so that the quotient $\mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} S_{(\psi, A)}$ inherits a \mathcal{G}_2 -action.

Lemma (II, 2.5). *Let $(\psi, A) \in \mathcal{C}_1$. The differential*

$$D_{(1,0)}\pi : L_2^2(M, i\mathbb{R}) \oplus S_{(\psi, A)} \longrightarrow L_1^2(M, E)$$

of π at the point $(1, 0) \in \mathcal{G}_2 \times S_{(\psi, A)}$ is surjective, and

$$\ker(D_{(1,0)}\pi) = \ker(G_{(\psi, A)}|_{L_2^2}) \oplus \{0\}.$$

Proof. The differential at the point $(1, 0)$ is given by

$$\begin{aligned} D_{(1,0)}\pi(f, \varphi, a) &= \left. \frac{d}{dt} \right|_{t=0} \exp(tf) \cdot ((\psi, A) + t(\varphi, a)) \\ &= G_{(\psi, A)}(f) + (\varphi, a). \end{aligned}$$

Hence, surjectivity of $D_{(1,0)}\pi$ is immediate from the decomposition

$$L_1^2(M, E) = \text{im}(G_{(\psi, A)}|_{L_2^2}) \oplus S_{(\psi, A)},$$

given in (II, 6). Moreover, as the above decomposition is direct,

$$G_{(\psi, A)}(f) + (\varphi, a) = 0 \iff G_{(\psi, A)}(f) = 0 \quad \text{and} \quad (\varphi, a) = 0$$

which proves the second assertion. \square

Proposition (II, 2.6). *For each $(\psi, A) \in \mathcal{C}_1$ there exists a $\mathcal{G}_{(\psi, A)}$ -invariant open neighbourhood V of $(1, 0)$ in $\mathcal{G}_2 \times S_{(\psi, A)}$ such that*

(i) $\pi|_V$ is a submersion.

(ii) The fibres of $\pi|_V$ are in 1-1 correspondence with the $\mathcal{G}_{(\psi, A)}$ -orbits.

Proof. We have to study two cases:

Case 1: (ψ, A) is irreducible: In this case $\ker(G_{(\psi, A)}) = 0$ so that according to the preceding lemma, the differential of π at $(1, 0)$ is an isomorphism. Invoking the inverse function theorem for Banach manifolds we conclude that there exists a neighbourhood V of $(1, 0)$ in $\mathcal{G}_2 \times S_{(\psi, A)}$ such that $\pi|_V$ is a diffeomorphism onto its image. As $\mathcal{G}_{(\psi, A)} = \{1\}$, the set V is clearly $\mathcal{G}_{(\psi, A)}$ -invariant.

Case 2: (ψ, A) is reducible, i.e., $\psi \equiv 0$: Lemma (II, 2.5) ensures that the differential of π at the point $(1, 0)$ is surjective. Invoking the implicit function theorem, we deduce that this holds true also on an open neighbourhood V of $(1, 0)$. Obviously, V can be chosen to be $\mathcal{G}_{(0, A)}$ -invariant since otherwise, we may consider $\mathcal{G}_{(0, A)} \cdot V$. Note that the differential of π is still surjective on that set because π is $\mathcal{G}_{(0, A)}$ -invariant.

It remains to prove the second assertion in this case. Suppose we have $(\gamma_i, \varphi_i, a_i) \in V$ such that $\pi(\gamma_1, \varphi_1, a_1) = \pi(\gamma_2, \varphi_2, a_2)$. Then, since $\psi = 0$,

$$(\gamma_1^{-1}\varphi_1, a_1 + 2\gamma_1^{-1}d\gamma_1) = (\gamma_2^{-1}\varphi_2, a_2 + 2\gamma_2^{-1}d\gamma_2).$$

Defining $\gamma := \gamma_2^{-1}\gamma_1$, we can express this alternatively as

$$\varphi_1 = \gamma^{-1}\varphi_2 \quad \text{and} \quad a_2 - a_1 = 2\gamma^{-1}d\gamma. \quad (\text{II, 8})$$

Then part (ii) is established provided that $\gamma \in \mathcal{G}_{(0, A)}$, i.e., that γ is constant.

Recall from (D, 15) that $[\gamma^{-1}d\gamma] \in H_{dR}^1(M; 2\pi i\mathbb{Z})$, which is a lattice in $H_{dR}^1(M; i\mathbb{R})$. If V is chosen small enough, cohomology classes of the form $[a_2 - a_1]$ can be forced to lie in a small neighbourhood of 0 in $H_{dR}^1(M; i\mathbb{R})$ hitting $H_{dR}^1(M; 2\pi i\mathbb{Z})$ only in 0. Hence, without loss of generality, the second part of (II, 8) can only be fulfilled if $[a_2 - a_1] = [\gamma^{-1}d\gamma] = 0$.

On the other hand, according to (II, 7), the 1-forms a_2 and a_1 are co-closed which implies that $2\gamma^{-1}d\gamma = a_2 - a_1$ is also co-closed and hence harmonic. Together with $[\gamma^{-1}d\gamma] = 0$, this implies $\gamma^{-1}d\gamma = 0$ and therefore, γ is constant. \square

After these preparations we can now state and prove the slice theorem. We follow the presentation in J.W. Morgan's book [42].

Theorem (II, 2.7). *Let (ψ, A) be an arbitrary configuration. Then there exists a $\mathcal{G}_{(\psi, A)}$ -invariant open neighbourhood U of $0 \in S_{(\psi, A)}$ such that*

$$\pi : \mathcal{G}_2 \times U \longrightarrow \mathcal{C}_1$$

induces a homeomorphism of $\mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} U$ onto a \mathcal{G}_2 -invariant open neighbourhood of (ψ, A) in \mathcal{C}_1 .

Proof. Let V be chosen as in Proposition (II, 2.6). Then $\mathcal{G}_2 \cdot V$ can be written as $\mathcal{G}_2 \times U$ where U is a $\mathcal{G}_{(\psi, A)}$ -invariant open neighbourhood of $0 \in S_{(\psi, A)}$. More concretely,

$$U := \{(\varphi, a) \mid \exists \gamma \in \mathcal{G}_2 : (\gamma, \varphi, a) \in V\}. \quad (\text{II, 9})$$

The map $\pi|_{\mathcal{G}_2 \times U}$ is a submersion since $\pi|_V$ is one and π is \mathcal{G}_2 -equivariant. Furthermore, $\pi(\mathcal{G}_2 \times U)$ is a \mathcal{G}_2 -invariant open neighbourhood of (ψ, A) in \mathcal{C}_1 .

We now establish the assertion by contradiction. Assume that possibly making V smaller, we cannot achieve that the induced map $\bar{\pi}$ on $\mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} U$ is injective. This means that for every V as in Proposition (II, 2.6) and corresponding U of the form (II, 9) there exists a point $(\gamma, \varphi, a) \in \mathcal{G}_2 \times U$ such that the fibre of $\pi|_{\mathcal{G}_2 \times U}$ which contains (γ, φ, a) is larger than the corresponding $\mathcal{G}_{(\psi, A)}$ orbit. We may thus choose sequences (φ_n, a_n) , (φ'_n, a'_n) in $S_{(\psi, A)}$ and (γ_n) in \mathcal{G}_2 such that

$$(\varphi_n, a_n) \xrightarrow{n \rightarrow \infty} 0, \quad (\varphi'_n, a'_n) \xrightarrow{n \rightarrow \infty} 0,$$

and

$$\gamma_n \cdot ((\psi, A) + (\varphi_n, a_n)) = (\psi, A) + (\varphi'_n, a'_n), \quad \text{but } \gamma_n \notin \mathcal{G}_{(\psi, A)}. \quad (\text{II, 10})$$

From this we conclude that

$$d\gamma_n = \frac{1}{2}\gamma_n(a'_n - a_n).$$

Since all γ_n are maps $M \rightarrow U_1$, the sequence (γ_n) is bounded with respect to $\|\cdot\|_\infty$. On the other hand, $(a'_n - a_n)$ converges to 0 in L_1^2 and thus also in L^p for all $p \leq 6$. Therefore, $(d\gamma_n)$ converges to 0 in every L^p for $p \leq 6$ which in turn provides an L_1^p -bound on (γ_n) . If $p > 3$, then there is a continuous multiplication $L_1^p \times L_1^2 \rightarrow L_1^2$. Invoking the above equation again shows that $(d\gamma_n)$ converges to 0 in L_1^2 . Consequently, (γ_n) is a bounded sequence in \mathcal{G}_2 .

We may thus deduce from the Rellich-Kondrachov Theorem (A, 1.2) that—possibly restricting to a subsequence—the sequence (γ_n) converges in $L_1^2(M, \mathbb{C})$. Let γ denote the limit point. Since $d\gamma_n \rightarrow 0$ in L^2 , we conclude that $d\gamma = 0$ weakly in L^2 . Since d is injectively elliptic on functions, regularity guarantees that $\gamma \in C^\infty(M, \mathbb{C})$ and that $d\gamma = 0$. In particular, γ is a constant function and (γ_n) converges to γ in L_1^2 . Actually, this convergence is with respect to L_2^2 because $d\gamma_n \rightarrow 0$ in L_1^2 and $d\gamma = 0$. By virtue of the embedding of L_2^2 in C^0 , this implies that γ takes values in U_1 since all γ_n do so. Invoking continuity of $\mathcal{G}_2 \times \mathcal{C}_1 \rightarrow \mathcal{C}_1$ we may now deduce from (II, 10) that

$$\gamma \cdot (\psi, A) = (\psi, A)$$

which shows that $\gamma \in \mathcal{G}_{(\psi, A)}$.

The remaining part of the proof works as in the finite dimensional case: Let us consider an open neighbourhood V of $(1, 0)$ in $\mathcal{G}_2 \times S_{(\psi, A)}$ as in Proposition (II, 2.6). Since V is $\mathcal{G}_{(\psi, A)}$ -invariant, we also have $(\gamma, 0) \in V$. As the sequences $(\gamma_n, \varphi_n, a_n)$ and $(1, \varphi'_n, a'_n)$ converge to $(\gamma, 0)$ and $(1, 0)$ respectively, there exists $n \in \mathbb{N}$ such that

$$(\gamma_n, \varphi_n, a_n) \in V \quad \text{and} \quad (1, \varphi'_n, a'_n) \in V$$

since V is an open neighbourhood of both limit points. By means of (II, 10),

$$\pi(\gamma_n, \varphi_n, a_n) = \pi(1, \varphi'_n, a'_n).$$

According to Proposition (II, 2.6), this requires $\gamma_n \in \mathcal{G}_{(\psi, A)}$ since the fibres of $\pi|_V$ correspond to the $\mathcal{G}_{(\psi, A)}$ orbits. However, $\gamma_n \in \mathcal{G}_{(\psi, A)}$ contradicts (II, 10).

We may thus suppose that the induced map $\bar{\pi}$ is injective on $\mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} U$. Moreover, since $\pi|_V$ is continuous and an open map, the same holds true for the respective restriction of $\bar{\pi}$. Hence, it induces a homeomorphism from $\mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} U$ onto the \mathcal{G}_2 -invariant open neighbourhood $\pi(\mathcal{G}_2 \times U)$ of (ψ, A) in \mathcal{C}_1 as is illustrated in the following diagram:

$$\begin{array}{ccc} \mathcal{G}_2 \times U & \xrightarrow{\pi} & \pi(\mathcal{G}_2 \times U) \\ \downarrow & \nearrow \bar{\pi} & \\ \mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} U & & \end{array}$$

□

Corollary (II, 2.8).

- (i) Suppose $U \subset S_{(\psi, A)}$ is chosen as in the slice theorem. Then $U/\mathcal{G}_{(\psi, A)}$ is homeomorphic to a neighbourhood of $[\psi, A]$ in $\mathcal{B}_1 = \mathcal{C}_1/\mathcal{G}_2$.

- (ii) *The irreducible part $\mathcal{B}_1^* \subset \mathcal{B}_1$ carries the structure of a smooth Banach manifold. Its tangent space at a point $[\psi, A]$ is naturally isomorphic to the local slice $S_{(\psi, A)}$.*
- (iii) *The projection $\mathcal{C}_1^* \rightarrow \mathcal{B}_1^*$ is a principal \mathcal{G}_2 -bundle.*

Proof. (i) Since \mathcal{G}_2 acts only on the first factor of $\mathcal{G}_2 \times U$, the quotient $(\mathcal{G}_2 \times U)/\mathcal{G}_2$ can be identified with U . Therefore, as the actions of \mathcal{G}_2 and $\mathcal{G}_{(\psi, A)}$ commute,

$$(\mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} U)/\mathcal{G}_2 \cong U/\mathcal{G}_{(\psi, A)}.$$

The map $\bar{\pi}$ is a \mathcal{G}_2 -equivariant homeomorphism hence induces a homeomorphism

$$(\mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} U)/\mathcal{G}_2 \cong \bar{\pi}(\mathcal{G}_2 \times_{\mathcal{G}_{(\psi, A)}} U)/\mathcal{G}_2.$$

This establishes part (i) for the right hand side is an open neighbourhood of $[\psi, A]$ in \mathcal{B}_1 .

- (ii) Let $(\psi, A) \in \mathcal{C}_1^*$ be an irreducible configuration, U a neighbourhood of $0 \in S_{(\psi, A)}$ as in the slice theorem. Suppose that $V := \pi(\mathcal{G}_2 \times U)$ is entirely contained in the irreducible part \mathcal{C}_1^* . We define a map $\Phi : V/\mathcal{G}_2 \rightarrow U$ by letting

$$\Phi([\gamma(\psi + \varphi, A + a)]) := (\varphi, a).$$

Note that Φ is well-defined, and that (i) ensures that it yields a homeomorphism $\mathcal{B}_1^* \supset V/\mathcal{G}_2 \cong U \subset S_{(\psi, A)}$. We have to ascertain that the collection of all such Φ provides \mathcal{B}_1^* with a differentiable structure.

Suppose that (ψ', A') is another irreducible configuration, and let $U' \subset S_{(\psi', A')}$ and $V' \subset \mathcal{C}_1^*$ be chosen correspondingly. Without loss of generality, we may assume that $V \cap V' \neq \emptyset$. Since V and V' are \mathcal{G}_2 -invariant, this yields

$$V/\mathcal{G}_2 \cap V'/\mathcal{G}_2 = (V \cap V')/\mathcal{G}_2 \neq \emptyset.$$

If $\tilde{U} := \Phi^{-1}(V/\mathcal{G}_2 \cap V'/\mathcal{G}_2)$ and $\tilde{U}' := \Phi'^{-1}(V/\mathcal{G}_2 \cap V'/\mathcal{G}_2)$, then the following diagram commutes.

$$\begin{array}{ccccc} \mathcal{G}_2 \times \tilde{U} & \xrightarrow{\pi} & V \cap V' & \xrightarrow{\pi'^{-1}} & \mathcal{G}_2 \times \tilde{U}' \\ \uparrow & & \downarrow & & \downarrow \\ \tilde{U} & \xrightarrow{\Phi^{-1}} & V/\mathcal{G}_2 \cap V'/\mathcal{G}_2 & \xrightarrow{\Phi'} & \tilde{U}' \end{array}$$

The slice theorem shows that π and π' are diffeomorphisms so that $\pi'^{-1} \circ \pi : \mathcal{G}_2 \times \tilde{U} \rightarrow \mathcal{G}_2 \times \tilde{U}'$ is smooth. Thus, the above diagram establishes that

$$\Phi' \circ \Phi^{-1} : \tilde{U} \subset S_{(\psi, A)} \rightarrow S_{(\psi', A')}$$

is also smooth. Therefore, the maps $\{\Phi : U \rightarrow S_{(\psi, A)}\}$ define a differentiable atlas of \mathcal{B}_1^* . This shows that \mathcal{B}_1^* is indeed a Banach manifold modelled on the isomorphism class of $S_{(\psi, A)}$.

(iii) This is an easy consequence of the proof of (ii). \square

Remark. The proof of (ii) shows that \mathcal{B}_1^* is actually a Hilbert manifold with respect to the induced L_1^2 scalar product on the local slice $S_{(\psi, A)}$. Whenever we use a scalar product on $S_{(\psi, A)}$, it shall, however, be the induced L^2 scalar product. To stress that the local slice is not complete with respect to $(\cdot, \cdot)_{L^2}$ we thus do not use the terminology *Hilbert* manifold.

3 Compactness

Our next task is to establish that the moduli space is sequentially compact. It will turn out that as corollary to the proof of the compactness theorem, the topology of the moduli space is independent of the initially chosen Sobolev orders.

Gauge fixing. The first idea leading to the results mentioned above is to find a suitable representative of a gauge equivalence class of monopoles. An appropriate method of fixing such a configuration is the so-called *Coulomb gauge*.

Lemma (II, 3.1). *Let $A, A_0 \in \mathcal{A}_1$. Then there exists $\gamma \in \mathcal{G}_2$ such that*

$$d^*(A - A_0 + 2\gamma^{-1}d\gamma) = 0,$$

i.e., each connection A is equivalent to a gauge field differing from a given A_0 only by a co-closed, imaginary valued 1-form.

Proof. The Hodge decomposition assures that

$$A - A_0 = \eta + df + d^*\omega,$$

where $f \in L_2^2(M, i\mathbb{R})$, $\omega \in L_2^2(M, \Lambda^2 iT^*M)$, and $\eta \in i\Omega(M)$ is harmonic. Let

$$\gamma := \exp\left(-\frac{f}{2}\right) \in L_2^2(M, U_1).$$

Then $2\gamma^{-1}d\gamma = -df$ and therefore,

$$d^*(A - A_0 + 2\gamma^{-1}d\gamma) = d^*(\eta + d^*\omega) = 0. \quad \square$$

Proposition (II, 3.2). *Let A_0 be a fixed gauge field of Sobolev class k_0 . If $(\psi, A) \in \mathcal{C}_1$ is a monopole such that $a := A - A_0$ is co-closed, then*

$$(\psi, A) \in L_{k_0}^2(M, S) \times \mathcal{A}_{k_0}.$$

Moreover, for all $k \in \mathbb{N}$ and $p \geq 2$ such that $L_{k_0}^2$ embeds in L_{k+1}^p , the following inequalities hold.

$$\begin{aligned} \|\psi\|_{L_{k+1}^p} &\leq \text{const} \cdot \left(\|c(a)\psi\|_{L_k^p} + \|\psi\|_{L_k^p} \right), \\ \|a\|_{L_{k+1}^p} &\leq \text{const} \cdot \left(\|q(\psi) - *F_{A_0}\|_{L_k^p} + \|Pa\|_{L_{k+1}^p} \right), \end{aligned} \quad (\text{II, 11})$$

where $P : L^2(M, iT^*M) \rightarrow L^2(M, iT^*M)$ denotes the L^2 -orthogonal projection onto the space of harmonic 1-forms.

Proof. The proof is an impressive application of the so-called *elliptic bootstrap* technique. Since $d^*a = 0$, we can reformulate the Seiberg-Witten equations for (ψ, A) in the following way

$$\begin{aligned} \mathcal{D}_{A_0}\psi &= -\frac{1}{2}c(a)\psi, \\ *(d + d^*)a &= \frac{1}{2}q(\psi) - *F_{A_0}. \end{aligned} \quad (\text{II, 12})$$

There is a Sobolev embedding of L_1^2 in L^6 . Therefore, a and ψ lie in L^6 and the Hölder inequality shows that $c(a)\psi \in L^3$. As \mathcal{D}_{A_0} is an elliptic operator, elliptic regularity for L^p Sobolev spaces applied to the first line of (II, 12) guarantees that $\psi \in L_1^3$. Hence, $\psi \in L^p$ for all $1 \leq p < \infty$. Employing the Hölder inequality again, we deduce that $q(\psi) \in L^p$ for all p .

Since $d + d^* : \Omega^* \rightarrow \Omega^*$ is elliptic, we obtain from elliptic regularity—this time applied to the second equation in (II, 12)—that $a \in L_1^p$ whenever $1 \leq p < \infty$. Proposition (A, 1.3) shows that there is a Sobolev multiplication $L_1^p \times L_1^3 \rightarrow L_1^2$ for all $p \geq 2$. Therefore, $c(a)\psi \in L_1^2$.

Again, ellipticity of \mathcal{D}_{A_0} yields $\psi \in L_2^2$. Applying Lemma (II, 1.1), we deduce that this yields $q(\psi) \in L_2^2$. Thus, elliptic regularity shows that $a \in L_3^2$. Using Sobolev multiplication and elliptic regularity in this manner further on we can prove inductively that $a \in L_k^2$ and $\psi \in L_k^2$ for all $1 \leq k \leq k_0$. Note that $*F_{A_0}$ is in $L_{k_0-1}^2$ since A_0 is of Sobolev class k_0 . Moreover, \mathcal{D}_{A_0} can be expressed as an elliptic operator with smooth coefficients plus a zero order term given by Clifford multiplication with a 1-form of Sobolev class k_0 , i.e.,

a continuous map $L_k^2(M, S) \rightarrow L_k^2(M, S)$ for all $1 \leq k \leq k_0$. Therefore, the bootstrapping does not cease at an earlier level.

Let $k \in \mathbb{N}$ and $p \geq 2$ such that $L_{k_0}^2$ embeds in L_{k+1}^p . Combining the elliptic estimate (A, 2), i.e.,

$$\|\psi\|_{L_{k+1}^p} \leq \text{const} \cdot (\|\mathcal{D}_{A_0}\psi\|_{L_k^p} + \|\psi\|_{L_k^p}),$$

with the first equation in (II, 12), we readily obtain the first inequality in (II, 11). Moreover, the triangle inequality yields

$$\|a\|_{L_{k+1}^p} \leq \|a - Pa\|_{L_{k+1}^p} + \|Pa\|_{L_{k+1}^p}.$$

The Poincaré inequality (A, 3) applied to $d + d^*$ shows that

$$\|a - Pa\|_{L_{k+1}^p} \leq \text{const} \cdot \|(d + d^*)(a - Pa)\|_{L_k^p} = \text{const} \cdot \|(d + d^*)a\|_{L_k^p}$$

because $(d + d^*)Pa = 0$. Inserting the second equation of (II, 12) in the above inequalities, we get the second inequality in (II, 11). \square

An immediate consequence of Proposition (II, 3.2) and Lemma (II, 3.1)—when combined with Sobolev embedding—is the following.

Corollary (II, 3.3). *Every SW-monopole is gauge equivalent to a C^∞ -monopole, i.e., every class in $\mathcal{M} \subset \mathcal{C}_1/\mathcal{G}_2$ has a representative which is smooth.*

The next task shall be to derive compactness of the moduli space from the inequalities (II, 11). For this we need to find a priori estimates for the maximum norms.

Lemma (II, 3.4). *Let $\mathcal{H}^1(M)$ denote the space of harmonic 1-forms and let $P : L^2(M, iT^*M) \rightarrow L^2(M, iT^*M)$ be the L^2 -orthogonal projection onto the subspace $i\mathcal{H}^1(M)$. Then there exists a constant $C > 0$ such that for each $a \in L^2(M, iT^*M)$ we can find $\gamma \in \mathcal{G}_2$ such that*

$$\|P(a + 2\gamma^{-1}d\gamma)\|_\infty < C.$$

Proof. The image of $H^1(M; \mathbb{Z})$ in $H^1(M; \mathbb{R})$ is a lattice. According to (D, 15), there is a surjective map

$$C^\infty(M, U_1) \rightarrow H_{dR}^1(M; 2\pi i\mathbb{Z}), \quad \gamma \mapsto [\gamma^{-1}d\gamma].$$

Therefore, the image of

$$C^\infty(M, U_1) \rightarrow i\mathcal{H}^1(M), \quad \gamma \mapsto P(2\gamma^{-1}d\gamma),$$

forms a lattice in the space of imaginary valued harmonic 1-forms. Given an arbitrary harmonic 1-form ω , we can thus find $\gamma \in C^\infty(M, U_1)$ such that

$$\|\omega + P(2\gamma^{-1}d\gamma)\|_\infty < C,$$

where C is a constant depending only on the lattice. Note that we are using that $\mathcal{H}^1(M)$ is finite dimensional. This implies the assertion. \square

The key estimate. The second ingredient to establish compactness of the moduli space is an a priori estimate for the norm of the spinor part of a monopole. We need the following result.

Lemma (II, 3.5). *Let ψ be a twice continuously differentiable spinor, for example, $\psi \in L^2_4(M, S)$. Then for every C^2 -gauge field A ,*

$$\Delta_g |\psi|^2 \leq 2 \operatorname{Re} \langle (\nabla^A)^* \nabla^A \psi, \psi \rangle,$$

where $\Delta_g := d^*d$ denotes the Laplacian of the Riemannian manifold (M, g) .

Proof. At an arbitrary point x , we choose a normal frame (e_1, e_2, e_3) with dual co-frame (e^1, e^2, e^3) . Using the fact that ∇^A is compatible with the metric, we deduce that at the point x ,

$$\begin{aligned} \Delta_g |\psi|^2 &= - * d * d \langle \psi, \psi \rangle = -2 * d \operatorname{Re} \langle \nabla_{e_i}^A \psi, \psi \rangle * e^i \\ &= -2 \operatorname{Re} \left(\langle \nabla_{e_j}^A \nabla_{e_i}^A \psi, \psi \rangle + \langle \nabla_{e_i}^A \psi, \nabla_{e_j}^A \psi \rangle \right) * (e^j \wedge * e^i) \\ &= -2 \sum_i \operatorname{Re} \langle \nabla_{e_i}^A \nabla_{e_i}^A \psi, \psi \rangle - 2 \sum_i |\nabla_{e_i}^A \psi|^2. \end{aligned}$$

Note that we have employed the relation $e^j \wedge * e^i = \delta^{ji} dv_g$. Recall that in a normal frame at x ,

$$(\nabla^A)^* \nabla^A \psi = - \sum_i \nabla_{e_i}^A \nabla_{e_i}^A \psi.$$

In combination with the above computations this implies the assertion. \square

Proposition (II, 3.6). *Suppose that (ψ, A) is a monopole which is at least C^2 . Then*

$$\|\psi\|_\infty^2 \leq \max \left\{ 0, \max_{x \in M} -2s_g(x) \right\} = \max \left\{ 0, -2 \min_{x \in M} s_g(x) \right\}, \quad (\text{II, 13})$$

where s_g denotes the scalar curvature of the Riemannian manifold (M, g) .

Proof. Combining Lemma (II, 3.5) with the Weitzenböck formula (D, 3.9), we obtain

$$\begin{aligned}\Delta_g|\psi|^2 &\leq 2 \cdot \operatorname{Re}\langle \mathcal{D}_A^2\psi, \psi \rangle - \frac{1}{2}s_g|\psi|^2 - \operatorname{Re}\langle c(F_A)\psi, \psi \rangle \\ &= -\frac{1}{2}s_g|\psi|^2 - \frac{1}{2}\operatorname{Re}\langle c(q(\psi))\psi, \psi \rangle,\end{aligned}$$

where we have employed the Seiberg-Witten equations in the last line.³ Using Proposition (I, 1.2), we infer that

$$\Delta_g|\psi|^2 \leq -\frac{1}{2}s_g|\psi|^2 - |q(\psi)|^2 = -\frac{1}{2}s_g|\psi|^2 - \frac{1}{4}|\psi|^4.$$

Let $x_0 \in M$ be a point where $|\psi|^2$ achieves its maximum. Then according to our sign convention,

$$\Delta_g|\psi|^2(x_0) \geq 0.$$

Together with the above estimate, this yields

$$\left(-\frac{1}{2}s_g|\psi|^2 - \frac{1}{4}|\psi|^4\right)(x_0) \geq 0.$$

We therefore obtain that

$$|\psi(x_0)|^2 = 0 \quad \text{or} \quad |\psi(x_0)|^2 \leq -2s_g(x_0).$$

Since $|\psi(x_0)|^2$ is maximal, this proves the proposition. \square

This result allows two immediate conclusions.

Corollary (II, 3.7). *Suppose (M, g) is a closed, oriented Riemannian 3-manifold whose scalar curvature is nonnegative, i.e., $s_g \geq 0$. Then every monopole (ψ, A) fulfills $\psi \equiv 0$. That is, the Seiberg-Witten moduli space consists only of equivalence classes of reducible configurations.*

Remark. This is a typical example of how gauge theory can be used to prove nonexistence results: As it shall turn out that the existence of irreducible monopoles is in a sense independent of the chosen Riemannian structure, finding an irreducible monopole with respect to an arbitrary metric prevents M from admitting a metric of nonnegative scalar curvature. Corresponding statements in the four dimensional case have turned out to be very useful. For a brief discussion of the above result's implications, we refer to Meng & Taubes [38].

³Note that we have also used that $c(*F_A) = c(F_A)$. This relation is an immediate consequence of our agreement to choose the spin representation in such a way that $c(dv_g) = -\operatorname{id}$. Recall that Clifford multiplication by 2-forms is defined via the isomorphism of vector spaces $\Lambda^\bullet V \cong \operatorname{Cl}(V)$.

Corollary (II, 3.8). *Let M be a closed, oriented Riemannian 3-manifold. Then there exist only finitely many spin^c structures on M for which the irreducible part of the moduli space is nonempty.*

Proof. Let σ be a spin^c structure on M with canonical class $c(\sigma)$. For any point in the moduli space we can find a representative (ψ, A) which is at least C^2 . According to Proposition (I, 1.2),

$$|q(\psi)| = \frac{1}{2}|\psi|^2,$$

so that the key estimate (II, 13) implies that

$$\psi = 0 \quad \text{or} \quad |q(\psi)| \leq -\min_{x \in M} s_g(x).$$

Since $*F_A = \frac{1}{2}q(\psi)$, this establishes a bound on $|F_A|$. According to the Chern-Weil construction, this implies that the image of $c(\sigma)$ in $H_{dR}^1(M; \mathbb{Z})$ lies in a bounded subset.

Therefore, only a finite subset of $H_{dR}^1(M; \mathbb{Z})$ corresponds to canonical classes of spin^c structures admitting irreducible monopoles. This proves the assertion since the number of canonical classes mapped to the same element in $H_{dR}^1(M; \mathbb{Z})$ is finite. Here, we are using that $H^2(M; \mathbb{Z})$ is finitely generated so that there cannot be infinitely many torsion elements. \square

Theorem (II, 3.9). *The SW-moduli space $\mathcal{M} \subset \mathcal{C}_1/\mathcal{G}_2$ is sequentially compact.*

Proof. We have to show that any sequence $([\psi_n, A_n])$ in \mathcal{M} contains a convergent subsequence. Choosing representatives $(\psi_n, A_n) \in \mathcal{C}_1$ and a fixed C^∞ -gauge field A_0 , we let $a_n := A_n - A_0$. As in Lemma (II, 3.4) we may apply gauge transformations to achieve that (Pa_n) is a bounded sequence in $i\mathcal{H}^1(M)$. Possibly gauge transforming again, we may also assume that $d^*a_n = 0$. Observe that the second property can be achieved using a gauge transformation of the form $\gamma := \exp(\frac{f}{2})$. Therefore, $2\gamma^{-1}d\gamma = df$ lies in the kernel of P and the fact that (Pa_n) is bounded remains unaffected. In consequence of the second condition, Proposition (II, 3.2) shows that all (ψ_n, A_n) are smooth configurations.

Step 1: For $p \geq 2$, the sequence (ψ_n, a_n) is bounded in $L_1^p(M, E)$:

Due to the key estimate (II, 13), the sequence (ψ_n) is bounded with respect to $\|\cdot\|_\infty$. Therefore, the sequence $(q(\psi_n) - *F_{A_0})_{n \geq 1}$ is also $\|\cdot\|_\infty$ -bounded and hence with respect to $\|\cdot\|_{L^p}$. Moreover, (Pa_n) is L_1^p -bounded because all norms are equivalent on the finite dimensional space $\mathcal{H}^1(M)$. Therefore,

the second equation in (II, 11) shows that (a_n) is a bounded sequence in L_1^p . Since (ψ_n) is bounded with respect to $\|\cdot\|_\infty$, we deduce that $(c(a_n)\psi_n)$ is bounded in L^p . Therefore, from the first inequality in (II, 11) we may infer that (ψ_n) is bounded in L_1^p .

Step 2: (ψ_n, a_n) is a bounded sequence in $L_2^2(M, E)$:

Proposition (A, 1.3) implies that if p is large enough (e.g. $p = 5$), there is a continuous multiplication

$$L_1^p \times L_1^p \rightarrow L_1^2.$$

As (ψ_n, a_n) is bounded in L_1^p for each $p \geq 2$, this shows that $(c(a_n)\psi_n)$ and $(q(\psi_n))$ are bounded with respect to $\|\cdot\|_{L_1^2}$. Since (Pa_n) is bounded in L_2^2 , the right hand sides of the inequalities (II, 11) are bounded. Therefore, (a_n) and (ψ_n) are bounded in L_2^2 .

The Rellich-Kondrachov Theorem (A, 1.2) shows that $L_2^2(M, E)$ embeds compactly in $L_1^2(M, E)$. Hence, there exists a subsequence of (ψ_n, A_n) which converges in \mathcal{C}_1 to, say, $(\psi, A) \in \mathcal{C}_1$. The Seiberg-Witten map $\text{SW} : \mathcal{C}_1 \rightarrow L^2(M, E)$ is continuous which yields $\text{SW}(\psi, A) = 0$. Therefore, $([\psi_n, A_n])$ contains a subsequence which converges in \mathcal{M} with respect to the induced topology. \square

Remark. A simple induction shows that the sequence (ψ_n, a_n) in the above proof is also bounded with respect to $\|\cdot\|_{L_k^2}$ for any $k \geq 1$: Assume that $k \geq 2$ and that (ψ_n, a_n) is bounded in L_k^2 . Then, since there is a continuous multiplication $L_k^2 \times L_k^2 \rightarrow L_k^2$, the sequences $c(a_n)\psi_n$ and $q(\psi_n)$ are also L_k^2 -bounded. The inequalities (II, 11) then guarantee that (ψ_n, a_n) is bounded in L_{k+1}^2 . We shall need this remark in the next paragraph.

Choosing different Sobolev orders. As was pointed out before, some authors endow the configuration space and the group of gauge transformations with different Sobolev structures. We now want to deduce from the above considerations that this does not affect the structure of the moduli space.

For any $k \geq 1$ we define

$$\mathcal{C}_k := L_k^2(M, S) \times \mathcal{A}_k \quad \text{and} \quad \mathcal{G}_{k+1} := L_{k+1}^2(M, \text{U}_1).$$

Since $k \geq 1$, we deduce from Example (A, 1.4) that there are continuous multiplications

$$L_{k+1}^2 \times L_{k+1}^2 \rightarrow L_{k+1}^2 \quad \text{and} \quad L_{k+1}^2 \times L_k^2 \rightarrow L_k^2.$$

This guarantees in the same way as before that \mathcal{G}_{k+1} is a Banach Lie group acting smoothly on \mathcal{C}_k . We define

$$\mathcal{B}_k := \mathcal{C}_k / \mathcal{G}_{k+1} \quad \text{and} \quad \mathcal{M}_k := (SW^{-1}(0) \cap \mathcal{C}_k) / \mathcal{G}_{k+1}.$$

Lemma (II, 3.10). *Let $(\psi, A) \in \mathcal{C}_k$ and $\gamma \in \mathcal{G}_2$ such that $\gamma \cdot (\psi, A) \in \mathcal{C}_k$. Then $\gamma \in \mathcal{G}_{k+1}$.*

Proof. The proof is another application of the elliptic bootstrap technique and is established by induction on k . If $k = 1$, the assertion is trivial. Hence, let $k \geq 2$ and assume that we have already proved that $\gamma \in \mathcal{G}_k$. As $A + 2\gamma^{-1}d\gamma \in \mathcal{A}_k$, we have

$$\gamma^{-1}d\gamma \in L_k^2(M, T^*M \otimes \mathbb{C}).$$

Since $k \geq 2$ there is a multiplication $L_k^2 \times L_k^2 \rightarrow L_k^2$ and we find

$$d\gamma = \gamma \cdot (\gamma^{-1}d\gamma) \in L_k^2(M, T^*M \otimes \mathbb{C})$$

From elliptic regularity we deduce that $\gamma \in L_{k+1}^2(M, \mathbb{C})$. □

We may now interpret \mathcal{B}_k as a subset of \mathcal{B}_1 : Since $\mathcal{C}_k \subset \mathcal{C}_1$, taking the quotient of \mathcal{C}_1 modulo \mathcal{G}_2 induces a map $\pi : \mathcal{C}_k \rightarrow \mathcal{B}_1$. From the above lemma we deduce that

$$\begin{aligned} \pi(\psi_1, A_1) = \pi(\psi_2, A_2) &\iff \exists_{\gamma \in \mathcal{G}_{k+1}} : (\psi_2, A_2) = \gamma \cdot (\psi_1, A_1) \\ &\iff [\psi_1, A_1] = [\psi_2, A_2] \text{ in } \mathcal{B}_k. \end{aligned}$$

Therefore, π induces an injective map

$$\bar{\pi} : \mathcal{B}_k \rightarrow \mathcal{B}_1.$$

Since $\mathcal{C}_k \subset \mathcal{C}_1$ is continuous, the map $\bar{\pi}$ is also continuous. In particular, \mathcal{B}_k is Hausdorff. Moreover, $\bar{\pi}$ restricts to an inclusion of the moduli spaces,

$$\mathcal{M}_k \hookrightarrow \mathcal{M}_1.$$

Corollary (II, 3.3) shows that this map is, in fact, a bijection. From the remark we stated after the proof of Theorem (II, 3.9), one easily establishes that \mathcal{M}_k is sequentially compact as well. Therefore, the inclusion of \mathcal{M}_k in \mathcal{M}_1 is a continuous bijection defined on a sequentially compact set. This implies continuity of the inverse as well, and we have the following result:

Corollary (II, 3.11). *The topology of the moduli space \mathcal{M} is independent of the chosen Sobolev orders, i.e., for any $k \geq 2$ the map*

$$\mathcal{M}_k \hookrightarrow \mathcal{M}_1$$

is a homeomorphism.

4 Local structure of the moduli space

We give a brief motivation for some considerations arising at this point in four dimensional Seiberg-Witten theory as well as in Yang-Mills theory. Consider the restriction of the Seiberg-Witten map to the irreducible part of \mathcal{C}_1 , i.e., $\text{SW} : \mathcal{C}_1^* \rightarrow L^2(M, E)$. We suppose that at some point $(\psi, A) \in \mathcal{C}_1^*$ the differential $F_{(\psi, A)}$ of SW is surjective. Under this assumption, the implicit function theorem ensures that the set of monopoles near (ψ, A) is a smooth manifold. Its tangent space at (ψ, A) is given by $\ker F_{(\psi, A)}$. Dividing out the group action and invoking the slice theorem shows that in this case a neighbourhood of $[\psi, A]$ in the moduli space is a smooth manifold which is modelled on the tangent space

$$T_{(\psi, A)}\mathcal{M} = \frac{\ker F_{(\psi, A)}}{\text{im}(G_{(\psi, A)}|_{L_2^2})}.$$

This space is the first cohomology group of the following complex:

$$0 \rightarrow L_2^2(M, i\mathbb{R}) \xrightarrow{G_{(\psi, A)}} L_1^2(M, E) \xrightarrow{F_{(\psi, A)}} L^2(M, E) \rightarrow 0.$$

In four dimensional Seiberg-Witten theory, the corresponding complex—being of a slightly different form than here—is elliptic so that the expected dimension of the moduli space can be computed as the index of rolled-up elliptic operator. At a first glimpse, the three dimensional situation is a bit more complicated since the above complex is not elliptic but has to be altered as in (I, 12). Nevertheless, by slightly reformulating the Seiberg-Witten equations and introducing “virtual” monopoles, the local analysis of the moduli space can be carried over exactly as in the four dimensional case (see Lim [32]). However, we shall take another approach since the arguments involved are more intuitive from a geometrical point of view.

The Chern-Simons-Dirac functional revisited. The nature of the Seiberg-Witten map as the gradient of the Chern-Simons-Dirac functional yields another possibility to analyse the local structure of the moduli space than the one via virtual monopoles mentioned above. This point of view reproduces the original⁴ ideas of Taubes [52] so that similarities with instanton theory on 3-manifolds become more intriguing. However, the essential ingredients—the slice theorem combined with the implicit function theorem—are the same in both approaches.

⁴A very clear explanation of Taubes’s ideas is given by P. Kirk in Sec. 3 of [24]. Our approach mimics the arguments given there.

For the time being, we restrict our attention to the irreducible part of \mathcal{B}_1 and consider the principal \mathcal{G}_2 -bundle $\mathcal{C}_1^* \rightarrow \mathcal{B}_1^*$. The assignment

$$(\psi, A) \mapsto S_{(\psi, A)} = \ker(G_{(\psi, A)}^*|_{L_1^2})$$

defines a smooth subbundle of the tangent bundle of \mathcal{C}_1^* . This follows from the fact that $G_{(\psi, A)}$ is injective with injectively elliptic first order term. Moreover, the bundle is \mathcal{G}_2 -invariant, i.e., for all $\gamma \in \mathcal{G}_2$,

$$\gamma \cdot \ker(G_{(\psi, A)}^*|_{L_1^2}) = \ker(G_{\gamma \cdot (\psi, A)}^*|_{L_1^2}).$$

From part (ii) of Corollary (II, 2.8) it becomes clear that this subbundle is the pullback of the tangent bundle of \mathcal{B}_1^* to \mathcal{C}_1^* . This in mind, we can relate objects defined on $T\mathcal{B}_1^*$ with objects on the tangent bundle of \mathcal{C}_1^* .

Lemma (II, 4.1). *The section SW is the pullback of the Chern-Simons-Dirac functional's L^2 -gradient on \mathcal{B}_1^* . In particular, the irreducible part of the moduli space is exactly the set of critical points of $\text{csd} : \mathcal{B}_1^* \rightarrow S^1$.*

Remark. As we have already noted, SW is only a gradient vector field in a weak sense since it takes values in the L^2 -completion of the tangent bundle of \mathcal{C}_1^* . Hence, we have also to consider the L^2 -completion of the tangent bundle of \mathcal{B}_1^* . According to Corollary (A, 2.8), the L^2 -completion of $\ker(G_{(\psi, A)}^*|_{L_1^2})$ coincides with $\ker G_{(\psi, A)}^*$.

Proof of Lemma (II, 4.1). We have to ascertain that SW is a \mathcal{G}_2 -equivariant map with values in the subbundle $\ker G^*$. The equivariance property has been proved earlier. For a smooth configuration (ψ, A) , we have

$$G_{(\psi, A)}^*(\text{SW}(\psi, A)) = 2d^*(\tfrac{1}{2}q(\psi) - *F_A) - i \text{Im}\langle \mathcal{D}_A \psi, \psi \rangle = 0$$

since Proposition (I, 2.1) implies that $d^*q(\psi) = i \text{Im}\langle \mathcal{D}_A \psi, \psi \rangle$. Since smooth configurations lie dense in \mathcal{C}_1 the assertion follows from continuity of SW. \square

The Hessian of $\text{csd} : \mathcal{C}_1^* \rightarrow \mathbb{R}$ at a point (ψ, A) is given by the differential $F_{(\psi, A)} : L_1^2(M, E) \rightarrow L^2(M, E)$ of SW. To obtain the pullback of the Hessian on \mathcal{B}_1^* we have to project $F_{(\psi, A)}$ to the subbundle $\ker G^*$ since this means taking the induced covariant derivative of the gradient $\mathcal{B}_1^* \rightarrow T\mathcal{B}_1^*$. Again, we have to account for the fact that $F_{(\psi, A)}$ maps $L_1^2(M, E)$ to $L^2(M, E)$. Then the pullback of Hessian of $\text{csd} : \mathcal{B}_1^* \rightarrow S^1$ is the unbounded operator in $\ker G_{(\psi, A)}^*$ given by

$$H_{(\psi, A)} := \text{Proj}_{\ker G^*} \circ F_{(\psi, A)}, \quad \text{dom}(H_{(\psi, A)}) := \ker(G_{(\psi, A)}^*|_{L_1^2}). \quad (\text{II, 14})$$

In consistency with the terminology in finite dimensional Morse theory, we now define:

Definition (II, 4.2). An irreducible Seiberg-Witten monopole is called *non-degenerate* if the Hessian is an invertible operator

$$H_{(\psi,A)} : \ker(G_{(\psi,A)}^*|_{L_1^2}) \rightarrow \ker G_{(\psi,A)}^*$$

Otherwise, it is called *degenerate*.

Clearly, this definition only depends on the gauge equivalence class of the monopole (ψ, A) . An immediate consequence of the inverse function theorem and Corollary (II, 2.8) is:

Proposition (II, 4.3). *Let $(\psi, A) \in \mathcal{C}_1^*$ be an irreducible, non-degenerate monopole. Then its gauge equivalence class $[\psi, A]$ is an isolated point of the moduli space $\mathcal{M}(\sigma)$.*

The Hessian $H_{(\psi,A)}$ is not very tractable since, for example, the Hilbert space in which it is defined depends on the point (ψ, A) . Moreover, it is not yet clear how to assign a “Hessian” to reducible configurations. Yet, as we are ultimately only interested in the spectral properties of $H_{(\psi,A)}$, we will relate the Hessian to the elliptic operator $T_{(\psi,A)}$ we considered at the end of Chapter I.

For the time being, we shall drop the reference to the base point (ψ, A) to simplify notation. We recall that,

$$T := (F + G, G^*) : L_1^2(M, E \oplus i\mathbb{R}) \rightarrow L^2(M, E \oplus i\mathbb{R}).$$

Hodge decomposition yields that

$$L_1^2(M, E) = \ker(G^*|_{L_1^2}) \oplus \operatorname{im}(G|_{L_2^2}) \quad \text{and} \quad L^2(M, E) = \ker G^* \oplus \operatorname{im} G. \quad (\text{II, 15})$$

With respect to this we now extend the Hessian H an operator in $L^2(M, E \oplus i\mathbb{R})$, with domain $L_1^2(M, E \oplus i\mathbb{R})$, by letting

$$\tilde{H} := \begin{pmatrix} H & 0 & 0 \\ 0 & 0 & G \\ 0 & G^* & 0 \end{pmatrix}. \quad (\text{II, 16})$$

Lemma (II, 4.4). *If (ψ, A) is an irreducible monopole, then $\tilde{H}_{(\psi,A)}$ coincides with the operator $T_{(\psi,A)}$. Moreover, (ψ, A) is non-degenerate if and only if $T_{(\psi,A)}$ is invertible.*

Proof. According to the definition of T and (II, 14), it suffices to show that the operator $F : L_1^2(M, E) \rightarrow L^2(M, E)$ satisfies

$$F = \text{Proj}_{\ker G^*} \circ F \circ \text{Proj}_{\ker G^*}. \quad (\text{II}, 17)$$

As we have noticed before, $F \circ G|_{L_2^2} = 0$ whenever (ψ, A) is a monopole. On the one hand this implies that $F = F \circ \text{Proj}_{\ker G^*}$ and on the other hand,

$$\text{im}(G|_{L_2^2}) \subset \ker F \subset \ker F^*$$

since F is symmetric. Hence, $\text{im } F \subset (\text{im } G)^\perp = \ker G^*$. Together, we get (II, 17). Then the second assertion follows from the next lemma. \square

Lemma (II, 4.5). *The eigenvalues of*

$$\tilde{G} := \begin{pmatrix} 0 & G \\ G^* & 0 \end{pmatrix} : \text{im}(G|_{L_2^2}) \oplus L_1^2(M, i\mathbb{R}) \longrightarrow \text{im } G \oplus L^2(M, i\mathbb{R}).$$

form a symmetric subset of \mathbb{R} , not containing 0. In particular, H is invertible if and only if \tilde{H} is.

Proof. Clearly, the operator \tilde{G} is symmetric with respect to the induced L^2 scalar product. Therefore, the set of eigenvalues is a subset of \mathbb{R} . If λ is an eigenvalue with corresponding eigenvector $((\varphi, a), f)$,

$$\tilde{G}((\varphi, a), -f) = -\lambda \cdot ((\varphi, a), -f).$$

Hence, $-\lambda$ is also an eigenvalue, and that shows the set of eigenvalues is symmetric. Moreover, G is injective since (ψ, A) is irreducible. This implies that \tilde{G} is injective as well. Since \tilde{H} is the direct sum of H and \tilde{G} , the second assertion follows. \square

Remark (II, 4.6). We will see in Corollary (II, 4.8) that \tilde{H} is a self-adjoint operator depending smoothly on (ψ, A) and having compact resolvent. Hence, given a C^1 -path of irreducible configuration, Definition (C, 1.9) of the spectral flow applies to the operators \tilde{H} associated to this path. Now, Lemma (II, 4.5) shows that the direct summand \tilde{G} gives no contribution to the spectral flow since the corresponding spectra are bounded away from 0. It is thus intuitively clear that, the spectral flow of the Hessian H may be represented by the spectral flow of \tilde{H} .

If $(\psi, A) \in \mathcal{C}_1^*$ is not a monopole, the equality $\tilde{H}_{(\psi, A)} = T_{(\psi, A)}$ does not necessarily hold. However, we can say the following:

Lemma (II, 4.7). *For each irreducible configuration (ψ, A) , the operator*

$$T_{(\psi, A)} - \tilde{H}_{(\psi, A)} : L_1^2(M, E \oplus i\mathbb{R}) \rightarrow L^2(M, E \oplus i\mathbb{R})$$

is compact and symmetric with respect to the L^2 scalar product.

Proof. It obviously suffices to show that, with respect to the decomposition (II, 15), the operator

$$F - \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix} : L_1^2(M, E) \rightarrow L^2(M, E)$$

is compact and symmetric. The latter property is obviously fulfilled since both, F and H , are symmetric. According to the definition of H , we are now to show that the operators

$$F|_{\text{im } G} : \text{im}(G|_{L_2^2}) \rightarrow L^2(M, E) \quad (\text{II, 18})$$

and

$$(F - \text{Proj}_{\ker G^*} \circ F)|_{\ker G^*} : \ker(G^*|_{L_1^2}) \rightarrow L^2(M, E) \quad (\text{II, 19})$$

are compact operators. Note that $G : L_2^2(M, i\mathbb{R}) \rightarrow \text{im}(G|_{L_2^2})$ is an isomorphism because it is injective and has closed range. Hence, compactness of (II, 18) is equivalent to compactness of

$$F \circ G|_{L_2^2} : L_2^2(M, i\mathbb{R}) \rightarrow L^2(M, E).$$

A short computation using part (iii) of Proposition (I, 1.2) shows that at the point (ψ, A) , for all $f \in L_2^2(M, i\mathbb{R})$,

$$F \circ G(f) = (-f \cdot \mathcal{D}_A \psi, 0).$$

If $p > 3$, there is a continuous Sobolev multiplication $L_1^p \times L^2 \rightarrow L^2$. On the other hand, L_2^2 embeds compactly in L_1^p for all $2 \leq p < 6$. This shows that $f \mapsto f \cdot \mathcal{D}_A \psi$ is a compact operator $L_2^2 \rightarrow L^2$. Thus, compactness of (II, 18) is proved. Regarding (II, 19), we claim that for $(\varphi, a) \in C^\infty(M, E)$,

$$(F - \text{Proj}_{\ker G^*} \circ F)(\varphi, a) = \text{Proj}_{\text{im } G} \circ F(\varphi, a) \in L_1^2(M, E). \quad (\text{II, 20})$$

For this note first, that $F(\varphi, a) \in L_1^2(M, E)$ since (φ, a) is smooth and F is a differential operator with L_1^2 coefficients. Second, projecting an element of $L_1^2(M, E)$ to $\text{im } G$ results in an element of $\text{im}(G|_{L_2^2})$ because G is injectively elliptic, cf. Corollary (A, 2.8). Together, this implies that (II, 20) holds for smooth (φ, a) . As L_1^2 embeds compactly in L^2 , this shows that the operator

$$\text{Proj}_{\text{im } G} \circ F : L_1^2(M, E) \rightarrow L^2(M, E),$$

restricted to the subspace $C^\infty(M, E)$ is compact. Since C^∞ is dense in L_1^2 , it follows that the operator is compact on the whole domain. This clearly implies compactness of (II, 19). \square

The above result shows that $\tilde{H}_{(\psi,A)}$ is obtained via a relative compact perturbation of $T_{(\psi,A)}$. Using the corresponding properties of $T_{(\psi,A)}$ (cf. Proposition (II, 1.5)), Theorem (A, 2.9) implies the following:

Corollary (II, 4.8). *Suppose (ψ, A) is an irreducible configuration. Then the operator $\tilde{H}_{(\psi,A)}$ defines a closed, self-adjoint Fredholm operator in $L^2(M, E \oplus i\mathbb{R})$. It has compact resolvent and thus discrete spectrum. Moreover, the assignment*

$$(\psi, A) \longmapsto \tilde{H}_{(\psi,A)}$$

is smooth with respect to the operator norm topology on \mathcal{L}_{sa} .

Corollary (II, 4.9). *Let $(\psi_t, A_t) : [a, b] \rightarrow \mathcal{C}_1^*$, be a C^1 -path of irreducible configurations such that (ψ_a, A_a) and (ψ_b, A_b) are monopoles. Then*

$$\text{SF}(T_{(\psi_a, A_a)}) = \text{SF}(\tilde{H}_{(\psi_t, A_t)}).$$

Proof. Consider the homotopy

$$[a, b] \times [0, 1] \rightarrow \mathcal{L}(L_1^2, L^2), \quad (t, s) \mapsto (1 - s) \cdot T_{(\psi_t, A_t)} + s \cdot \tilde{H}_{(\psi_t, A_t)},$$

which—due to our assumption—leaves the endpoints fixed. It follows as in Corollary (II, 4.8) that this homotopy takes values in the space \mathcal{L}_{sa} of closed, self-adjoint operators in L^2 with domain L_1^2 . Thus, according to Proposition (C, 1.10), the paths $T_{(\psi_t, A_t)}$ and $\tilde{H}_{(\psi_t, A_t)}$ have the same spectral flow. \square

Reducible configurations. Until now we have restricted our attention to irreducible configurations for the above geometrical motivation is only meaningful on the manifold \mathcal{B}_1^* . However, $T_{(\psi,A)}$ is defined independently of ψ being zero or not. If A is a reducible configuration, then the explicit formula is

$$T_{(0,A)} = \begin{pmatrix} \mathcal{D}_A & 0 & 0 \\ 0 & -*d & 2d \\ 0 & 2d^* & 0 \end{pmatrix}. \quad (\text{II, 21})$$

Therefore,

$$\begin{aligned} \ker T_{(0,A)} &= \ker \mathcal{D}_A \oplus \ker(d \oplus d^*) \oplus \ker d \\ &= \ker \mathcal{D}_A \oplus H_{dR}^1(M; i\mathbb{R}) \oplus H_{dR}^0(M; i\mathbb{R}). \end{aligned} \quad (\text{II, 22})$$

Proof of (II, 22). From (II, 21) it is clear that $T_{(0,A)}(\varphi, a, f) = 0$ if and only if $\mathcal{D}_A \varphi = 0$, $2df = *da$, and $d^*a = 0$. According to the Hodge decomposition of $L^2(M, T^*M)$, we have $\text{im } d \perp \text{im } d^*$. Therefore, since $*da = -d^*(a)$, this implies that $2df = *da$ if and only if both, df and da , vanish. \square

Remark. We shall see in the next chapter that if A is a reducible monopole, then the summand $H_{dR}^1(M; i\mathbb{R})$ of $\ker T_{(0,A)}$ represents the “tangent space” to the reducible part of the moduli space whereas $\ker \mathcal{D}_A$ determines whether $[0, A]$ is an accumulation point for irreducible elements.

5 Counting monopoles

We shall now equip the points of \mathcal{B}_1 with a sign. With respect to this, the algebraic count of points in the moduli space will be the number which lies in the center of our interest. Clearly, this is only meaningful if the number of gauge equivalence classes of monopoles is finite. Yet, compactness of the moduli space and the fact that non-degenerate monopoles lie isolated in the irreducible part of \mathcal{B}_1 suggest that this might indeed be true—at least modulo some small perturbation of the Chern-Simons-Dirac functional. How this perturbation has to be chosen is the topic of Section III.2. For the time being we shall have to assume that the irreducible part of the moduli space, $\mathcal{M}^*(\sigma)$, consists only of non-degenerate monopoles and is finite.

Another expression of the algebraic count of monopoles, which we shall derive shortly after its initial definition, is reminiscent of a kind of Euler characteristic associated to \mathcal{B}_1^* . The subject of Chapter III is, essentially, to prove that this number is indeed independent of the chosen Riemannian metric (and the perturbation term), thus yielding a smooth invariant of the underlying 3-manifold M .

After these preliminary remarks let us now equip each configuration with a sign. Recall from Proposition (II, 1.6) that the assignment $(\psi, A) \mapsto T_{(\psi,A)}$ is a smooth map from \mathcal{C}_1 to \mathcal{L}_{sa} , the space of self-adjoint operators in L^2 with domain L_1^2 . If $(\psi_t, A_t) : [a, b] \rightarrow \mathcal{C}_1$ is a continuous path of configurations, then the associated family $\{T_{(\psi_t, A_t)}\}_{t \in [a, b]}$ also depends continuously on t and we are in the situation of Definition (C, 2.2).

Definition (II, 5.1). For each configuration (ψ, A) let

$$\varepsilon(\psi, A) := \varepsilon(T_{(t\psi, A)}; 0 \leq t \leq 1)$$

be the orientation transport along the family $\{T_{(t\psi, A)}\}_{t \in [0, 1]}$ assigned to the affine path from $(0, A)$ to (ψ, A) .

Lemma (II, 5.2). *For every configuration (ψ, A) the following holds:*

- (i) *If A_0 is an arbitrary connection, then*

$$\varepsilon(\psi, A) = (-1)^{\text{SF}(T_{(0, A_0)} + t(\psi, A - A_0))}.$$

(ii) For every gauge transformation $\gamma \in \mathcal{G}$,

$$\varepsilon(\psi, A) = \varepsilon(\gamma \cdot (\psi, A)).$$

Proof. (i) The linear path from $(0, A)$ to (ψ, A) is homotopic to the path going from $(0, A)$ to $(0, A_0)$ and then to (ψ, A) . Using the homotopy invariance and the additivity property of the spectral flow, we deduce that

$$\text{SF}(T_{(t\psi, A)}) = \text{SF}(T_{(0, A) + t(0, A_0 - A)}) + \text{SF}(T_{(0, A_0) + t(\psi_0, A - A_0)}). \quad (\text{II}, 23)$$

According to (II, 21), the first path entering the right hand side is the direct sum of a path of complex linear operators and a constant path. Since we are regarding $L^2(M, S)$ as an \mathbb{R} -Hilbert space, the spectral flow of a path of complex linear operators is always congruent 0 mod 2. Moreover, the spectral flow of a constant path is 0 so that the first summand in (II, 23) is 0 mod 2. Now, using Theorem (C, 2.5) and inserting the considerations we just made we find that

$$\varepsilon(\psi, A) = (-1)^{\text{SF}(T_{(t\psi, A)})} = (-1)^{\text{SF}(T_{(0, A_0) + t(\psi_0, A - A_0)})}.$$

(ii) From the equivariance of SW one readily deduces that T is \mathcal{G}_2 -equivariant which means that for all $(\varphi, a, f) \in L_1^2(M, E \oplus i\mathbb{R})$,

$$T_{\gamma \cdot (\psi, A)}(\varphi, a, f) = \gamma \cdot T_{(\psi, A)}(\gamma^{-1} \cdot (\varphi, a, f)).$$

Recall for this from (I, 6) that \mathcal{G}_2 acts only on the spinor part of $E \oplus i\mathbb{R}$. Using this, one straightforwardly concludes that the determinant line bundles of $T_{(t\psi, A)}$ and $T_{\gamma \cdot (t\psi, A)}$ are canonically isomorphic via the isomorphism induced by

$$\ker T_{(t\psi, A)} \rightarrow \ker T_{\gamma \cdot (t\psi, A)}, \quad (\varphi, a, f) \mapsto \gamma \cdot (\varphi, a, f)$$

Then it is immediate that the orientation transports along $T_{(t\psi, A)}$ and $T_{\gamma \cdot (t\psi, A)}$ coincide. \square

Definition (II, 5.3). Assume that $\mathcal{M}^*(\sigma)$ consists only of non-degenerate monopoles and is finite. Then we let

$$\text{sw}_0(\sigma) := \sum_{[\psi, A] \in \mathcal{M}^*(\sigma)} \varepsilon(\psi, A), \quad (\text{II}, 24)$$

where (ψ, A) is an arbitrary representative of $[\psi, A]$.

Note that as a consequence of Lemma (II, 5.2), part (ii), the number sw_0 is well-defined. Moreover, Corollary (II, 3.8) immediately implies that $\text{sw}_0(\sigma)$ vanishes for all but finitely many spin^c structures σ .

Morse theoretical interpretation. From part (i) of Lemma (II, 5.2) we now deduce a geometrical motivation for the definition of $\text{sw}_0(\sigma)$. For this let us briefly recall the ideas we need from finite dimensional Morse theory:

If $f : M \rightarrow \mathbb{R}$ is a Morse function on a compact manifold, then the Euler characteristic of M can be expressed as the signed count of critical points. Here, the sign associated to a critical point is given by the parity of its Morse index, i.e., the number of negative eigenvalues of the respective Hessian.

However, this expression of the Euler characteristic does not necessarily require an a priori knowledge of all Morse indices. If we fix one critical point x_0 , then the Morse index of another critical point x can be computed via the Morse index at the point x_0 and the *difference* of the Morse index of x relative to the index of x_0 . In other words, what we need is an understanding of how the number of negative eigenvalues of the Hessian changes from one critical point to another, i.e., we have to consider the spectral flow of the Hessian along paths connecting two critical points. In this way, the problem of having to define an “index” for operators with unbounded spectrum in the infinite dimensional setting at hand can be overcome. These ideas in mind, we now give an alternative expression of $\text{sw}_0(\sigma)$.

Proposition (II, 5.4). *Assume that $\mathcal{M}^*(\sigma)$ is finite and consists only of gauge equivalence classes of non-degenerate monopoles. Then for every fixed $(\psi_0, A_0) \in \mathcal{M}^*(\sigma)$,*

$$\text{sw}_0(\sigma) = \varepsilon(\psi_0, A_0) \cdot \sum_{[\psi, A] \in \mathcal{M}^*(\sigma)} (-1)^{\text{SF}(T_{(\psi_0, A_0) + t(\psi - \psi_0, A - A_0)})}. \quad (\text{II, 25})$$

Proof. The affine path connecting $(0, A_0)$ with (ψ, A) is homotopic to the concatenation of the affine path from $(0, A_0)$ to (ψ_0, A_0) and the affine path from (ψ_0, A_0) to (ψ, A) . Hence,

$$\text{SF}(T_{(0, A_0) + t(\psi, A - A_0)}) = \text{SF}(T_{t\psi_0, A_0}) + \text{SF}(T_{(\psi_0, A_0) + t(\psi - \psi_0, A - A_0)}).$$

Inserting this in Lemma (II, 5.2), part (i), we immediately get the assertion. \square

Remark.

- (i) Corollary (II, 4.9) and Remark (II, 4.6) show that the fact that (II, 25) involves the operator T rather than the Hessian H does not collide with the ideas of the Morse theoretical motivation.

- (ii) The term $\varepsilon(\psi_0, A_0)$ occurring in formula (II, 25) corresponds to fixing the parity of one particular Morse index. Without this term only the absolute value of $\text{sw}_0(\sigma)$ could be expected to be an invariant. The sign convention of Chen [11] is a bit more complicated but seems to be the same we have chosen. The description in terms of $\varepsilon(\psi, A)$ exhibits a natural choice for this convention—depending, however, on the way of how to define the spectral flow of a path whose endpoints are not invertible.

Chapter III

Seiberg-Witten Invariants of 3-Manifolds

Until now there is one major problem in the definition of $\text{sw}_0(\sigma)$: We cannot guarantee that all critical points of the Chern-Simons-Dirac functional are non-degenerate. As in finite dimensional Morse theory we cannot expect that the signed count of critical points describes the Euler characteristic if there also exist degenerate ones. Therefore, we are lead to study perturbations of the Chern-Simons-Dirac functional in order to obtain non-degenerateness. This is explicitly carried out in Section 2. Before, we include a section about the abstract setting which lies behind these ideas.

In the remaining parts of this chapter we will then analyse the behaviour of $\text{sw}_0(\sigma)$ under deformation of the Riemannian metric and the perturbation. It turns out that for manifolds with $b_1 > 1$ we achieve topological invariance of the signed count of monopoles in this way. If $b_1 \leq 1$, a topological invariant can also be obtained—at least with some extra effort.

1 Regular values and perturbed level sets

Suppose that $\Phi : X \rightarrow Y$ is a smooth Fredholm map between paracompact Banach manifolds, i.e., Φ is a smooth map such that for every $x \in X$, the differential $D_x\Phi : T_xX \rightarrow T_{\Phi(x)}Y$ is a Fredholm operator. If X is connected—what we will henceforth assume—then the function $x \mapsto \text{ind } D_x\Phi$ is constant on X . We thus can define the *index* of Φ as this common value.

Let us assume for a moment that $y_0 \in Y$ is a regular value of Φ , i.e., $D_x\Phi$ is surjective for every $x \in M := \Phi^{-1}(y_0)$. Then M is either empty or a smooth submanifold of X with $\dim M = \text{ind } \Phi$. In the applications we have in mind, this assumption is usually not fulfilled. The following considerations

show how to achieve regularity by perturbing Φ .

Suppose that P is an affine Banach space, modelled on a separable Banach space E . Let Φ extend to a C^m -map $\hat{\Phi} : X \times P \rightarrow Y$ with $\hat{\Phi}(\cdot, p_0) = \Phi$ for some $p_0 \in P$. We will usually refer to P as the *perturbation space* and call $\hat{\Phi}$ the *perturbation map*. In addition, we require that $\hat{\Phi}(\cdot, p) : X \rightarrow Y$ is Fredholm for each $p \in P$. As P is connected, the index of $\hat{\Phi}(\cdot, p)$ remains to be equal to the index of Φ as p varies. Moreover, a reasonable perturbation has to be chosen in such a way that y_0 is a regular value of $\hat{\Phi}$, which we will assume in the following. Then the level set

$$\widehat{M} := \hat{\Phi}^{-1}(y_0)$$

is either empty or a (possibly infinite dimensional) C^m -submanifold of $X \times P$.

Proposition (III, 1.1). *The projection map $\pi : \widehat{M} \rightarrow P$ is a C^m -Fredholm map, with index equal to $\text{ind } \Phi$. Thus for each regular value p of π , the set $M_p := \pi^{-1}(p)$ defines an $\text{ind } \Phi$ -dimensional C^m -submanifold of X . Moreover, p is a regular value of π if and only if y_0 is a regular value of the map $\hat{\Phi}(\cdot, p)$.*

Proof. The asserted differentiability properties are obvious so that we have only to compute the index of the differential at a point $(x, p) \in \widehat{M}$. Consider the maps $\hat{\Phi}_p := \hat{\Phi}(\cdot, p) : X \rightarrow Y$ and $\hat{\Phi}_x := \hat{\Phi}(x, \cdot) : P \rightarrow Y$ obtained by fixing p and x respectively. Clearly, if we prove that there exist algebraic isomorphisms

$$\ker D_{(x,p)}\pi \cong \ker D_x(\hat{\Phi}_p) \quad \text{and} \quad \text{coker } D_{(x,p)}\pi \cong \text{coker } D_x(\hat{\Phi}_p), \quad (\text{III, 1})$$

the Fredholm property of π and the assertion about its index are immediate. To prove (III, 1) note first of all that the tangent space of \widehat{M} at (x, p) is given by

$$T_{(x,p)}\widehat{M} = \ker D_{(x,p)}\hat{\Phi} = \ker (D_p(\hat{\Phi}_x) + D_x(\hat{\Phi}_p)). \quad (\text{III, 2})$$

From this the first equation of (III, 1) follows because

$$\begin{aligned} \ker D_x(\hat{\Phi}_p) &\cong \{(v, 0) \in T_x X \oplus T_p P \mid D_x(\hat{\Phi}_p)(v) = 0\} \\ &= \{(v, w) \in T_{(x,p)}\widehat{M} \mid w = 0\} = \ker D_{(x,p)}\pi. \end{aligned}$$

Next observe that (III, 2) implies

$$\begin{aligned} \text{im } D_{(x,p)}\pi &= \{w \in T_p P \mid \exists_{v \in T_x X} : (v, w) \in T_{(x,p)}\widehat{M}\} \\ &= \{w \in T_p P \mid D_p(\hat{\Phi}_x)(w) \in \text{im } D_x(\hat{\Phi}_p)\}. \end{aligned}$$

Hence, there is an algebraic isomorphism

$$\frac{\operatorname{im} D_{(x,p)}\pi}{\ker D_p(\hat{\Phi}_x)} \cong \operatorname{im} D_p(\hat{\Phi}_x) \cap \operatorname{im} D_x(\hat{\Phi}_p).$$

Furthermore, there always exists an abstract isomorphism

$$\frac{T_p P}{\ker D_p(\hat{\Phi}_x)} \cong \operatorname{im} D_p(\hat{\Phi}_x).$$

Together, these isomorphisms imply that

$$\frac{T_p P}{\operatorname{im} D_{(x,p)}\pi} \cong \frac{\left(\frac{T_p P}{\ker D_p(\hat{\Phi}_x)}\right)}{\left(\frac{\operatorname{im} D_{(x,p)}\pi}{\ker D_p(\hat{\Phi}_x)}\right)} \cong \frac{\operatorname{im} D_p(\hat{\Phi}_x)}{\operatorname{im} D_p(\hat{\Phi}_x) \cap \operatorname{im} D_x(\hat{\Phi}_p)}.$$

Invoking surjectivity of $D_{(x,p)}\hat{\Phi}$, we can decompose $T_{y_0}Y$ as

$$T_{y_0}Y = \operatorname{im} D_p(\hat{\Phi}_x) + \operatorname{im} D_x(\hat{\Phi}_p) \cong \frac{\operatorname{im} D_p(\hat{\Phi}_x)}{\operatorname{im} D_p(\hat{\Phi}_x) \cap \operatorname{im} D_x(\hat{\Phi}_p)} \oplus \operatorname{im} D_x(\hat{\Phi}_p).$$

Finally, we get the following chain of isomorphisms

$$\operatorname{coker} D_{(x,p)}\pi = \frac{T_p P}{\operatorname{im} D_{(x,p)}\pi} \cong \frac{\operatorname{im} D_p(\hat{\Phi}_x)}{\operatorname{im} D_p(\hat{\Phi}_x) \cap \operatorname{im} D_x(\hat{\Phi}_p)} \cong \operatorname{coker} D_x(\hat{\Phi}_p)$$

This proves the second part of (III, 1). Hence, $\pi : \widehat{M} \rightarrow P$ is a Fredholm map of index $\operatorname{ind} D_x(\hat{\Phi}_p)$ at the point (x, p) . By our assumptions this index equals the index of Φ .

Suppose now that p is a regular value of π . Then according to the above, $\operatorname{coker} D_x(\hat{\Phi}_{p_0}) = 0$ for all $x \in M_p$. Hence, $D_x(\hat{\Phi}_p)$ is surjective whenever $x \in M_p$. This shows that y_0 is a regular value of $\hat{\Phi}_p$. \square

The importance of the above proposition becomes obvious when we combine it with an infinite dimensional version of Sard's Theorem due to Smale. Recall that a subset of a topological space is called *generic* (or equivalently *of second category*) if it is the countable intersection of open and dense subsets.

Theorem (III, 1.2). (cf. [51], Thm. 1.3). *Let $\pi : M \rightarrow P$ be a C^m -Fredholm map between two paracompact Banach manifolds, and suppose that $m > \max\{0, \operatorname{ind} \pi\}$. Then the set of regular values of π is a generic subset of P . In particular, for a generic choice of $p \in P$, the level set $\pi^{-1}(p)$ is either empty or a C^m -submanifold of M with dimension equal to $\operatorname{ind} \pi$.*

The well-known Baire category Theorem states that a generic subset of a complete metric space is necessarily dense. We can thus combine Theorem (III, 1.2) with Proposition (III, 1.1) to find arbitrarily small perturbations such that the level sets are regular.

Parametrized level sets. We now consider a family of C^m -Fredholm maps $\{\Phi^g : X \rightarrow Y\}$ depending smoothly on an additional parameter $g \in R$, where R denotes a connected Banach manifold. In our applications—where for example, R is a completion of the space of all Riemannian metrics on a fixed 3-manifold—we want to compare the level sets $M(g) := (\Phi^g)^{-1}(y_0)$ for different parameters. For this let g_1 and g_{-1} be two distinct elements of R and let $g = g_t : [-1, 1] \rightarrow R$ be a smooth path connecting them. Then the corresponding level sets at the endpoints are related by a cobordism given by the y_0 -level set of

$$\Psi : X \times [-1, 1] \rightarrow Y, \quad \Psi(x, t) := \Phi^{g_t}(x). \quad (\text{III, 3})$$

Since

$$D_{(x,t)}\Psi = D_x(\Phi^{g_t}) + D_t(\Phi^g(x)), \quad (\text{III, 4})$$

the map Ψ is a C^m -Fredholm map of index $1 + \text{ind } \Phi$. This is because the first summand of (III, 4) is a Fredholm operator $T_x X \oplus \mathbb{R} \rightarrow T_{\Psi(x,t)} Y$ of index $1 + \text{ind } \Phi^{g_t}$, whereas the second term is rank 1 so that it does not affect the Fredholm index.

As in the preceding paragraph, the idea is now that we get regularity of the y_0 -level set of Ψ if we consider small perturbations. For this we consider a smooth family of perturbation maps

$$\hat{\Phi}^g : X \times P \rightarrow Y, \quad g \in R,$$

such that each $\hat{\Phi}^g$ is a C^m -Fredholm map with regular value y_0 . Applying the Sard-Smale Theorem to the projections $(\hat{\Phi}^{g_i})^{-1}(y_0) \rightarrow P$, we find parameters p_1 and p_{-1} in P such that y_0 is a regular value of $\hat{\Phi}_{p_i}^{g_i} := \hat{\Phi}^{g_i}(\cdot, p_i)$ for both $i \in \{-1, 1\}$. To relate the corresponding level sets

$$M_{p_i}(g_i) := (\hat{\Phi}_{p_i}^{g_i})^{-1}(y_0), \quad i \in \{-1, 1\},$$

by a regular cobordism we define a new perturbation space by letting

$$\hat{P} := \{p : [-1, 1] \rightarrow P \mid p \text{ is } C^m \text{ and } p(i) = p_i \text{ for } i \in \{-1, 1\}\}.$$

Since P is an affine space, modelled on some separable Banach space E the set \hat{P} is also an affine space, modelled on the Banach space

$$\{w : [-1, 1] \rightarrow E \mid w \text{ is } C^m \text{ and } w(i) = 0 \text{ for } i \in \{-1, 1\}\}.$$

Note that this is indeed a Banach space with respect to uniform convergence of all derivatives of w up to order m . In the spirit of the last paragraph the map Ψ given in (III, 3) is now perturbed by

$$\hat{\Psi} : X \times [-1, 1] \times \hat{P} \rightarrow Y, \quad \hat{\Psi}(x, t, p) := \hat{\Phi}^{g_t}(x, p_t).$$

To apply Proposition (III, 1.1) we have to ascertain the following:

Lemma (III, 1.3). *Let $\hat{\Psi}$ be defined as above. Then the point y_0 is a regular value of $\hat{\Psi}$.*

Proof. Let $(x, t, p) \in \hat{\Psi}^{-1}(y_0)$ and consider

$$D_{(x,t,p)}\hat{\Psi}(v, s, w) = D_{(x,p_t)}(\hat{\Phi}^{g_t})(v, w_t) + s \cdot \frac{d}{dt}\bigg|_t \hat{\Phi}^{g_t}(x, p_t).$$

Since y_0 is a regular value of $\hat{\Phi}^{g_t}$ for any t , the first operator on the right hand side is surjective. Therefore, (x, p, t) is a regular point of $\hat{\Psi}$ provided that (v, w_t) can be chosen arbitrarily. This is, however, only true if $t \notin \{-1, 1\}$ since w is subject to the condition $w_i = 0$ for $i \in \{-1, 1\}$. But in this case,

$$D_{(x,p_i)}(\hat{\Phi}^{g_i})(v, 0) = D_x(\Phi_{p_i}^{g_i})(v)$$

which is already surjective due to the choice of p_{-1} and p_1 . \square

As a consequence of Proposition (III, 1.1), the projection map

$$\Pi : \hat{\Psi}^{-1}(y_0) \longrightarrow \hat{P}$$

is a C^m -Fredholm map with index equal to $1 + \text{ind } \Phi$. We thus can apply Sard-Smale again, deducing that for a generic path $p = p_t \in \hat{P}$, the parametrized level set

$$\widehat{M}_p(g) := \Pi^{-1}(p) \cong \bigcup_{t \in [-1, 1]} (M_{p_t}(g_t) \times \{t\}) \subset X \times [-1, 1]$$

is either empty or an $(1 + \text{ind } \Phi)$ -dimensional C^m -submanifold of $X \times [-1, 1]$. Summarizing the above considerations, we have proved:

Proposition (III, 1.4). *Let $g = g_t : [-1, 1] \rightarrow R$ be a smooth path. Suppose that p_{-1} and p_1 are chosen in such a way that y_0 is a regular value of $\hat{\Phi}_{p_i}^{g_i}$ for $i \in \{-1, 1\}$. Then for a generic path $p = p_t \in \hat{P}$, the parametrized level set*

$$\widehat{M}_p(g) := \bigcup_{t \in [-1, 1]} (M_{p_t}(g_t) \times \{t\}) \subset X \times [-1, 1]$$

is either empty or a C^m -submanifold of dimension $(1 + \text{ind } \Phi)$.

Remark (III, 1.5). As was pointed out before, we apply the preceding results to the L^2 -gradient vector field of the Chern-Simons-Dirac functional. Therefore, we have to generalize the above considerations slightly as the gradient vector field is not a Fredholm map between Banach manifolds but a *Fredholm section* of a bundle of Banach spaces, i.e., a section which in local trivializations can be represented by Fredholm maps. We will now describe the changes to be made:

Given a smooth Fredholm section $\Phi : X \rightarrow V$ of a bundle of Banach spaces $V \rightarrow X$, where X denotes a connected, paracompact Banach manifold, consider the intrinsic differential $D_x\Phi : T_xX \rightarrow V_x$. If this is surjective at any zero x of Φ , then the section Φ is called *transversal to the zero section*. In this case, the zero set $\Phi^{-1}(0)$ is a smooth submanifold of X of dimension equal to $\text{ind } \Phi$. As before, we might have to perturb Φ to obtain transversality. Similarly, this is done by studying an extension map $\hat{\Phi}$, which is in this case a section of the pullback bundle $\text{pr}_1^*V \rightarrow X \times P$, where P is some perturbation space. If we can achieve that $\hat{\Phi}$ is transversal to the zero section of $\text{pr}_1^*V \rightarrow X \times P$, then the union of all perturbed zero sets $\widehat{M} := \hat{\Phi}^{-1}(0)$ is a smooth submanifold of $X \times P$, and Proposition (III, 1.1) and Prop.(III, 1.4) continue to hold also in this setting.

2 The perturbed Seiberg-Witten equations

Using the preceding section as a guideline, we now consider perturbations of the Chern-Simons-Dirac functional to obtain transversality of the Seiberg-Witten map. To put it another way, we want to produce a moduli space consisting solely of non-degenerate critical points. Moreover, the perturbed moduli space should preferably contain only irreducible points. As we shall see, there are topological obstructions to the latter which lie encoded in the first Betti number.

To begin with, we have to specify an appropriate set of perturbations. Let $Z_k^2(M; i\mathbb{R})$ denote the space of pure imaginary valued, closed¹ 2-forms of some fixed Sobolev class $k \geq 2$.

Definition (III, 2.1). Let (M, g) be a closed, oriented Riemannian 3-manifold equipped with a spin^c structure σ . For $\eta \in Z_k^2(M; i\mathbb{R})$ we define

¹Many authors choose co-closed 1-forms of an appropriate Sobolev class as perturbations which is in more agreement with 4-dimensional Seiberg-Witten theory. We follow Lim's approach in [32] since the space of closed 2-forms does not depend on the metric which is of some convenience later.

the η -perturbed Chern-Simons-Dirac functional by

$$\begin{aligned} \text{csd}_\eta(\psi, A) &:= \text{csd}(\psi, A) + \int_M (A - A_0) \wedge \eta \\ &= \frac{1}{2} \int_M \left(\langle \psi, \mathcal{D}_A \psi \rangle dv_g + (A - A_0) \wedge (F_A + F_{A_0} + 2\eta) \right), \end{aligned}$$

where A_0 is a fixed element of $\mathcal{A}(\sigma)$.

Using the computations at the end of Chapter I as a pattern, one finds that the L^2 -gradient of csd_η is given by²

$$\begin{aligned} \text{SW}_\eta : \mathcal{C}(\sigma) &\longrightarrow L^2(M, E) \\ (\psi, A) &\longmapsto \text{SW}(\psi, A) - (0, *\eta). \end{aligned} \tag{III, 5}$$

Definition (III, 2.2). Let (M, g) be a closed, oriented Riemannian 3-manifold equipped with a spin^c structure σ . For $\eta \in Z_k^2(M; i\mathbb{R})$ we call $(\psi, A) \in \mathcal{C}(\sigma)$ an η -monopole if it is a critical point of csd_η , i.e., if it solves the η -perturbed Seiberg-Witten equations:

$$\boxed{\begin{aligned} \mathcal{D}_A \psi &= 0 \\ *(F_A + \eta) &= \frac{1}{2} q(\psi) \end{aligned}}. \tag{III, 6}$$

The moduli space of η -monopoles modulo gauge equivalence is denoted by $\mathcal{M}_\eta(\sigma) \subset \mathcal{B}(\sigma)$.

Note that $\mathcal{M}_\eta(\sigma)$ is well-defined since SW_η is \mathcal{G} -equivariant. This is because the group of gauge transformations acts only on the spinor part of $L^2(M, E)$ so that equivariance of SW_η follows from equivariance of SW . Moreover, the section SW_η restricts to a section of the bundle $\ker G^* \rightarrow \mathcal{C}^*$ because η is closed and therefore,

$$G_{(\psi, A)}^* \circ \text{SW}_\eta(\psi, A) = G_{(\psi, A)}^* \circ \text{SW}(\psi, A) - 2d^*(\eta) = 0.$$

As in Section II.4, we shall usually work equivariantly on the bundle $\ker G^* \rightarrow \mathcal{C}^*$ instead of regarding SW_η as an L^2 -gradient vector field on \mathcal{B}^* . In this context, the covariant derivative of $\text{SW}_\eta : \mathcal{C} \rightarrow \ker G^*$ is again the index 0 Fredholm operator

$$H_{(\psi, A)} : \ker(G_{(\psi, A)}^*|_{L_1^2}) \rightarrow \ker G_{(\psi, A)}^*$$

²In this chapter we are going to drop the reference to the Sobolev class of configurations and gauge transformations, i.e., we write \mathcal{C} instead of \mathcal{C} and so on.

as defined in (II, 14). For this note that the dependence on η vanishes when we differentiate SW_η .

Regularity of the perturbed moduli space. Following the guideline of Section 1, our next task is to establish that the perturbation map

$$\widehat{\text{SW}} : \mathcal{C}^* \times Z_k^2(M; i\mathbb{R}) \rightarrow \ker G^*, \quad \widehat{\text{SW}}((\psi, A), \eta) := \text{SW}_\eta(\psi, A)$$

is transversal to the zero section. It is straightforward to see that at a point $((\psi, A), \eta)$, its derivative is given by the index 1 Fredholm operator

$$\begin{aligned} \ker(G_{(\psi, A)}^*|_{L_1^2}) \oplus Z_k^2(M; i\mathbb{R}) &\rightarrow \ker G_{(\psi, A)}^* \\ ((\varphi, a), \nu) &\mapsto H_{(\psi, A)}(\varphi, a) + (0, - * \nu). \end{aligned} \quad (\text{III, 7})$$

Proposition (III, 2.3). *The perturbation map $\widehat{\text{SW}}$ is transversal to the zero section.*

Proof. Suppose that $((\psi, A), \eta)$ is a zero of $\widehat{\text{SW}}$. Since we are interested only in gauge equivalence classes of η -monopoles, we may assume that modulo a gauge transformation, the η -monopole (ψ, A) is at least of Sobolev class k (see Remark (III, 2.6) below). Under this assumption we now have to show that the derivative of $\widehat{\text{SW}}$ at $((\psi, A), \eta)$ is surjective. Note that the derivative has closed image as it is Fredholm. Thus we consider $(\varphi, a) \in \ker G^*$, orthogonal to the image, wanting to show that this implies $(\varphi, a) = 0$.

For each (φ, a) which is orthogonal to the image of (III, 7), we have

$$(\varphi, a) \in (\text{im } H)^\perp = \ker H \quad \text{and} \quad a \perp *Z_k^2(M; i\mathbb{R}), \quad (\text{III, 8})$$

where we are using that H is self-adjoint and are dropping the reference to (ψ, A) . From the definition of H it is immediate that

$$\ker H = \ker F \cap \ker G^* = \ker T \cap \ker G^*,$$

where F is the differential of SW and T is defined in (I, 13). This shows that

$$(\varphi, a) \in \ker F \quad \text{and} \quad (\varphi, a) \in L_k^2(M, E) \quad (\text{III, 9})$$

because (ψ, A) is of Sobolev class k so that the operator T is elliptic with L_k^2 -coefficients and we may thus apply elliptic regularity. Therefore, φ and a are at least continuous as $k \geq 2$.

From (III, 8) we deduce that a must be closed. Together with the fact that $(\varphi, a) \in \ker F$, this yields

$$0 = q(\psi, \varphi) - *da = q(\psi, \varphi). \quad (\text{III, 10})$$

Let U be the open subset of M on which ψ is nowhere vanishing. Invoking Proposition (I, 1.2), we infer from (III, 10) that there exists $f \in L_k^2(U, i\mathbb{R})$ such that

$$\varphi|_U = f\psi|_U.$$

Note that L_k^2 -regularity of f follows from the explicit formula

$$f = i|\psi|^{-2}\langle\varphi, i\psi\rangle.$$

To proceed, we require more information about the structure of U . This is provided by the so-called *unique continuation principle* which we state without proof.

Theorem. (cf. [8], Thm. 8.2). *Let M be a compact Riemannian manifold and S a $\text{Cl}(M)$ -module with $\text{Cl}(M)$ -compatible connection. Then the unique continuation principle is valid for the corresponding Dirac operator \mathcal{D} . That is, any solution φ of $\mathcal{D}\varphi = 0$ which vanishes on an open subset of M also vanishes on the whole connected component of M .*

As we have chosen (ψ, A) irreducible, the spinor ψ is nonzero so that U is nonempty. Furthermore, ψ is harmonic with respect to \mathcal{D}_A . Hence, the unique continuation principle ensures that U is dense since the complement of U cannot contain any open subset of M .

Using the first part of the equation $F(\varphi, a) = 0$, we find that

$$0 = \mathcal{D}_A(f\psi|_U) + \frac{1}{2}c(a)\psi|_U = f\mathcal{D}_A\psi|_U + c(df + \frac{1}{2}a)\psi|_U.$$

Therefore, since $\mathcal{D}_A\psi = 0$ and as $\psi|_U$ is nowhere vanishing,

$$df + \frac{1}{2}a|_U = 0. \quad (\text{III, 11})$$

Now $(\varphi, a) \in \ker G^*$ ensures that

$$0 = G^*(\varphi|_U, a|_U) = G^*(f\psi|_U, a|_U) = 2d^*a|_U - i\text{Im}\langle f\psi|_U, \psi|_U\rangle.$$

Applying d^* to (III, 11) then yields

$$0 = 4d^*df + 2d^*a|_U = 4d^*df + i\text{Im}\langle f\psi|_U, \psi|_U\rangle = 4d^*df + |(\psi|_U)|^2 f.$$

Multiplying this equation with f and integrating over U results in

$$0 = \int_U \left(4\langle df, df \rangle + |\psi|^2 f^2 \right) dv_g.$$

For $\psi|_U$ is nowhere vanishing, we deduce that $f = 0$ and thus also $\varphi|_U = 0$. In view of (III, 11), we hence obtain that φ and a vanish on an open and dense subset of M . Because of continuity, we can finally draw the conclusion that $(\varphi, a) = 0$. \square

As a corollary to Proposition (III, 1.1) and to Theorem (III, 1.2) we now obtain from the above result:

Theorem (III, 2.4). *Let (M, g) be a closed, oriented Riemannian 3-manifold with spin^c structure σ . Then, for a generic choice of $\eta \in Z_k^2(M; i\mathbb{R})$, the irreducible part of the η -perturbed moduli space consists of non-degenerate points.*

Introducing a perturbation does not change the basic topological features of the moduli space. As it is a topological subspace of $\mathcal{B}(\sigma)$, the set $\mathcal{M}_\eta(\sigma)$ is Hausdorff. Modifying the proof of the key estimate (II, 13) slightly, one readily gets the following:

Proposition (III, 2.5). *Let $\eta \in Z_k^2(M; i\mathbb{R})$, where $k \geq 4$. Suppose (ψ, A) is an η -monopole of some Sobolev class ≥ 4 . Then*

$$\|\psi\|_\infty^2 \leq \max \left\{ 0, -2 \min_{x \in M} s_g(x) + 2\|\eta\|_\infty \right\}. \quad (\text{III, 12})$$

This proposition implies compactness of $\mathcal{M}_\eta(\sigma)$ in the same way as in Section II.3.

Remark (III, 2.6). It should be pointed out that in contrast to Chapter II the η -perturbed moduli spaces need not consist of gauge equivalence classes of smooth configurations. Reviewing the corresponding proofs shows that we can only guarantee regularity up to the Sobolev order of η .

Reducible η -monopoles. Compactness of $\mathcal{M}_\eta(\sigma)$ in combination with Theorem (III, 2.4) does not necessarily imply that the irreducible part of $\mathcal{M}_\eta(\sigma)$ is a finite set. This is because some *reducible* η -monopole might be an accumulation point. We thus have to understand the structure of the reducible locus.

Proposition (III, 2.7). *Let M be a closed, oriented 3-manifold with spin^c structure σ . Then, for every perturbation $\eta \in Z_k^2(M; i\mathbb{R})$, there exist reducible η -monopoles if and only if the cohomology class of η satisfies*

$$2\pi ic(\sigma) = [\eta], \quad (\text{III, 13})$$

where $c(\sigma) \in H_{dR}^1(M; \mathbb{R})$ denotes the canonical class of σ . If nonempty, the set of gauge equivalence classes of reducible η -monopoles is homeomorphic to the b_1 -dimensional torus

$$H_{dR}^1(M; i\mathbb{R}) / H_{dR}^1(M; 4\pi i\mathbb{Z}),$$

where b_1 denotes the first Betti number of M .

Proof. Suppose that there exists a reducible η -monopole A . Then $*(F_A + \eta)$ is zero, and this implies that

$$2\pi ic(\sigma) = 2\pi i[\frac{i}{2\pi}F_A] = 2\pi i[\frac{1}{2\pi i}\eta] = [\eta].$$

On the other hand, $2\pi ic(\sigma) = [\eta]$ can only hold if $[F_{A_0}] = -[\eta]$ for each gauge field A_0 . This means that there exists an imaginary valued 1-form a such that $-\eta = F_{A_0} + da$. We conclude that $A_0 + a$ is a reducible η -monopole.

Let $Z_1^1(M; i\mathbb{R})$ be the space of closed, imaginary valued 1-forms of Sobolev class 1. Supposing for the rest of the proof that the set of such elements is nonempty, we fix a reducible η -monopole A_0 . Then the map

$$\Phi : Z_1^1(M; i\mathbb{R}) \rightarrow \mathcal{C}, \quad \Phi(a) := A_0 + a,$$

parametrizes the whole set of reducible η -monopoles. As we have seen before, two elements of the form $A_0 + a$ and $A_0 + a + df$ lie in the same gauge orbit which shows that the map Φ descends to a continuous map

$$\bar{\Phi} : H_{dR}^1(M; i\mathbb{R}) \rightarrow \mathcal{M}_\eta(\sigma).$$

The image of $\bar{\Phi}$ consists of the reducible part of $\mathcal{M}_\eta(\sigma)$. It remains to insure that

$$\bar{\Phi}([a]) = \hat{\Phi}([a']) \iff [a' - a] \in \text{im} (H_{dR}^1(M; 4\pi i\mathbb{Z}) \rightarrow H_{dR}^1(M; \mathbb{R})).$$

This is, however, only a restatement of the considerations in (D, 15). Hence, $\bar{\Phi}$ factors to a bijective and continuous map between the b_1 -dimensional torus $H_{dR}^1(M; i\mathbb{R})/H_{dR}^1(M; 4\pi i\mathbb{Z})$ and the reducible part of the moduli space. Since the torus is compact, this map is necessarily a homeomorphism. \square

An immediate conclusion of this result shall play a major role later: If $b_1 = 0$, condition (III, 13) is always fulfilled. Since a 0-dimensional torus is simply a point, we get:

Corollary (III, 2.8). *Let M be a closed, oriented 3-manifold with vanishing first Betti number, and let σ be a spin^c structure on M . Then, for every perturbation $\eta \in Z_k^2(M; i\mathbb{R})$, there exists exactly one reducible point in $\mathcal{M}_\eta(\sigma)$.*

Suitable perturbations on manifolds with $b_1 > 0$. Since we want to achieve that the perturbed moduli space consists only of irreducible points, Proposition (III, 2.7) naturally leads to considering the restricted perturbation space

$$\mathcal{P}_k(\sigma) := \{\eta \in Z_k^2(M; i\mathbb{R}) \mid 2\pi ic(\sigma) \neq [\eta]\}. \quad (\text{III, 14})$$

As we have seen above the restricted perturbation space is empty if $b_1 = 0$. Yet, if $b_1 > 0$, then $\mathcal{P}_k(\sigma)$ is an open, dense subset of $Z_k^2(M; i\mathbb{R})$.

Definition (III, 2.9). Let (M, g) be a closed and oriented Riemannian 3-manifold with spin^c structure σ and first Betti number $b_1 > 0$.

- (i) An element $\eta \in \mathcal{P}_k(\sigma)$ is called a *suitable perturbation* with respect to g if the η -perturbed moduli space $\mathcal{M}_\eta(\sigma; g)$ consists only of finitely many non-degenerate, irreducible points.
- (ii) If $\eta \in \mathcal{P}_k(\sigma)$ is a suitable perturbation with respect to g , we define

$$\text{sw}_\eta(\sigma; g) := \sum_{[\psi, A] \in \mathcal{M}_\eta(\sigma; g)} \varepsilon(\psi, A).$$

Recall that $\varepsilon(\psi, A)$ is defined as the orientation transport along the family $T_{(t\psi, A)}$ associated to the linear path connecting $(0, A)$ to (ψ, A) . Moreover, as an immediate consequence of Theorem (III, 2.4) we obtain that the set of suitable perturbations is a generic subset of $Z_k^2(M; i\mathbb{R})$. In particular, there exist suitable perturbation with respect to arbitrary metrics on M .

Remark. In Section 5 we will return to the question of how to choose perturbations appropriately in the remaining case $b_1 = 0$.

3 Invariance for manifolds with $b_1 > 1$

We now want to prove that on 3-manifolds with $b_1 > 1$, the number $\text{sw}_\eta(\sigma; g)$ is independent of the metric g and the perturbation η .

The parametrized moduli space. Before we take up the proof of the theorem we have in mind, we make some general observations concerning the perturbed moduli spaces associated to two different metrics g_{-1} and g_1 .

Consider a Sobolev completion of the space of Riemannian metrics and let $\{g_t\}_{t \in [-1, 1]}$ be a continuous family of metrics. For every continuous path of perturbations, $\eta_t : [-1, 1] \rightarrow Z_k^2(M; i\mathbb{R})$, we define the *parametrized moduli space* by

$$\widehat{\mathcal{M}}_\eta(\sigma; g) := \bigcup_{t \in [-1, 1]} \mathcal{M}_{\eta_t}(\sigma; g_t) \times \{t\} \subset \mathcal{B} \times [-1, 1]. \quad (\text{III, 15})$$

Remark. Note that this definition necessitates a procedure to identify the spaces $\mathcal{B}(\sigma; g_t)$ for different parameters t . The material we need for this is

summarized in Section D.4. We choose, say, g_0 as a fixed reference metric and redefine $\mathcal{M}_{\eta_t}(\sigma; g_t)$ as the zero set (modulo gauge equivalence) of the map

$$(\psi, A) \mapsto \text{SW}_{\eta_t}^{g_t}(\psi, A) := (\mathcal{D}_A^t \psi, \frac{1}{2} q_t(\psi) - *_t(F_A + \eta_t)),$$

where $*_t$ denotes the Hodge-star-operator on T^*M induced by the metric g_t . Moreover, we employ the notation of Section D.4 and write

$$\mathcal{D}_A^t \psi := \hat{\kappa}_t^{-1} \mathcal{D}_A^{g_t} \hat{\kappa}_t \psi \quad \text{and} \quad q_t(\psi) := q^{g_t}(\hat{\kappa}_t \psi),$$

where $\hat{\kappa}_t : L^2(M, S; g_0) \rightarrow L^2(M, S; g_t)$ is induced by identifying the spinor bundles associated to different metrics.

Irrespective of the value of b_1 , the following holds:

Proposition (III, 3.1). *Let M be a closed, oriented 3-manifold with spin^c structure σ , and let $\{g_t\}_{t \in [-1, 1]}$ be a continuous family of Riemannian metrics on M . Then, for every continuous path $\eta : [-1, 1] \rightarrow Z_k^2(M; i\mathbb{R})$, the parametrized moduli space $\widehat{\mathcal{M}}_\eta(\sigma; g) \subset \mathcal{B}(\sigma) \times [-1, 1]$ is sequentially compact.*

Proof. Let $([\psi_n, A_n, t_n])_{n \geq 1}$ be a sequence in $\widehat{\mathcal{M}}_\eta(\sigma; g)$. Without loss of generality, we can assume that t_n converges to some $t_0 \in [-1, 1]$. We then have to show that there exists a subsequence of $([\psi_n, A_n])_{n \geq 1}$ which converges to an element in $\mathcal{M}_{\eta_{t_0}}(\sigma; g_{t_0})$. Essentially, this amounts to transferring the corresponding arguments in the proof of the compactness theorem of Chapter II. We shall only give a brief sketch of how this is done.

We choose g_{t_0} as a reference metric. As before, we may represent $[\psi_n, A_n]$ by configurations $(\psi_n, A_0 + a_n)$, where a_n satisfies $d^{*_{t_0}} a_n = 0$. Reviewing the proof of Proposition (II, 3.2), we draw the conclusion that this yields $(\psi_n, a_n) \in L_k^2(M, E)$, where $k \geq 2$ is the Sobolev class of η_t . Moreover, we can achieve—by possibly invoking another gauge transformation—that the sequence of the harmonic parts of a_n (with respect to g_{t_0}) is bounded. The elements (ψ_n, a_n) are solutions to

$$\begin{aligned} \mathcal{D}_{A_0}^{t_n} \psi_n &= -\frac{1}{2} c(\hat{\kappa}_{t_n}^{-1} a_n) \psi_n \\ (d + d^{*_{t_n}}) a_n &= *_t \frac{1}{2} q_{t_n}(\psi_n) - F_{A_0} - \eta_{t_n} \end{aligned}$$

Since $(\hat{\kappa}_{t_n} \psi_n, A_0 + a_n)$ is an η_{t_n} -monopole (with respect to the metric g_{t_n}), we deduce as in Chapter II that the $L^2(g_{t_n})$ -norm of the second equation's left hand side is smaller than some number depending on the scalar curvature s_{t_n} and the g_{t_n} -norm of η_{t_n} . Here, we have to use the estimate (III, 12) instead of the key estimate in Chapter II. Since g_t and η_t are continuous paths, this data depends continuously on the parameter t . We

thus also obtain a $L^2(g_{t_0})$ -bound on $(\ast_{t_n} \frac{1}{2} q_{t_n}(\psi_n) - F_{A_0} - \eta_{t_n})$. Arguing exactly as before, we deduce that a_n is bounded with respect to $L_1^2(g_{t_0})$. Proceeding in this manner, we can finally achieve that (ψ_n, a_n) is a bounded sequence with respect to the $L_2^2(g_{t_0})$ -norm, and this implies the assertion. \square

The Seiberg-Witten invariant. We are now in the position to prove the main result of this thesis. Our presentation follows the proof of the corresponding result in the four dimensional case as given in Nicolaescu's book [45], Sec. 2.3.2.

Theorem (III, 3.2). *Let M be a closed, oriented 3-manifold with spin structure σ and first Betti number $b_1 > 1$. Suppose that g_{-1} and g_1 are two Riemannian metrics on M and that η_{-1} and η_1 are respectively chosen suitable perturbations. Then*

$$\text{sw}_{\eta_{-1}}(\sigma; g_{-1}) = \text{sw}_{\eta_1}(\sigma; g_1).$$

We thus can define the “Seiberg-Witten invariant” of the spin^c manifold (M, σ) by letting

$$\text{sw}(\sigma) := \text{sw}_{\eta}(\sigma; g),$$

where g is an arbitrary Riemannian metric, and η is a suitable perturbation with respect to g .

Proof. Let $\widehat{\mathcal{P}}_k(\sigma)$ denote the space of all continuously differentiable paths $\eta : [-1, 1] \rightarrow \mathcal{P}_k(\sigma)$ which connect η_{-1} and η_1 . We endow this space with its natural C^1 -topology thus providing it with the structure of a Banach manifold. Since we are using only the restricted perturbation space, the parametrized moduli space associated to a path $\eta \in \widehat{\mathcal{P}}_k(\sigma)$ is entirely contained in the irreducible part $\mathcal{B}^* \times [-1, 1]$. For all η in a certain generic subset of $\widehat{\mathcal{P}}_k(\sigma)$, Proposition (III, 1.4) implies that $\widehat{\mathcal{M}}_{\eta}(\sigma; g)$ is either empty or carries the structure of a 1-dimensional C^1 -submanifold of $\mathcal{B}^* \times [-1, 1]$. In the first case it is immediate that

$$\text{sw}_{\eta_{-1}}(\sigma; g_{-1}) = \text{sw}_{\eta_1}(\sigma; g_1) = 0.$$

We shall thus assume from now on that the parametrized moduli space is nonempty, hence a 1-dimensional manifold. According to Proposition (III, 3.1) this manifold is sequentially compact. Moreover, its boundary is given by

$$(\mathcal{M}_{\eta_{-1}}(\sigma; g_{-1}) \times \{-1\}) \cup (\mathcal{M}_{\eta_1}(\sigma; g_1) \times \{1\}).$$

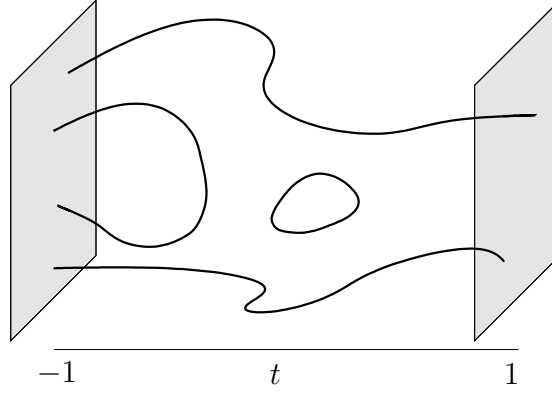


Figure III.1: The parametrized moduli space

Therefore, $\widehat{\mathcal{M}}_\eta(\sigma; g)$ consists of a finite union of continuously differentiable arcs, say, $c_i : [a_i, b_i] \rightarrow \mathcal{B}^* \times [-1, 1]$, $i = 1, \dots, n$, whose endpoints lie on the boundary (see Fig. III.1). Note that we are neglecting closed arcs since they neither contribute to $\text{sw}_{\eta-1}(\sigma; g_{-1})$ nor to $\text{sw}_{\eta_1}(\sigma; g_1)$.

To simplify notation we want to embed $\widehat{\mathcal{M}}_\eta(\sigma; g)$ in $\mathcal{C}^* \times [-1, 1]$. This can be done in the following way: We choose representatives for $c_i(a_i)$ and lift the paths $c_i : [a_i, b_i] \rightarrow \mathcal{B}^* \times [-1, 1]$ horizontally to $\mathcal{C}^* \times [-1, 1]$. This can be done since $\mathcal{C}^* \rightarrow \mathcal{B}^*$ is a principal bundle with $\ker G^* \rightarrow \mathcal{C}^*$ as a horizontal structure. Then the tangent space to $\widehat{\mathcal{M}}_\eta(\sigma; g)$ at a point (ψ, A, t) is given by the kernel of

$$D_{(\psi, A, t)} \widehat{\text{SW}}_\eta : \ker(G_{(\psi, A)}^*|_{L_1^2}) \oplus \mathbb{R} \rightarrow \ker G_{(\psi, A)}^*,$$

i.e., it is given by all $(\varphi, a, x) \in \ker(G_{(\psi, A)}^*|_{L_1^2}) \oplus \mathbb{R}$ such that

$$H_{(\psi, A)}^{g_t}(\varphi, a) + x \cdot \frac{d}{dt} \Big|_t \text{SW}_{\eta_t}^{g_t}(\psi, A) = 0, \quad (\text{III, 16})$$

where $H_{(\psi, A)}^{g_t}$ denotes the Hessian at (ψ, A) with respect to the metric g_t . To relate this with the operator T , we infer from Lemma (II, 4.4) that for each zero of $\widehat{\text{SW}}_\eta$,

$$T_{(\psi, A, t)} := T_{(\psi, A)}^{g_t} = \begin{pmatrix} H_{(\psi, A)}^{g_t} & 0 & 0 \\ 0 & 0 & G_{(\psi, A)} \\ 0 & G_{(\psi, A)}^{*t} & 0 \end{pmatrix},$$

where we are using the decomposition $(\ker G^* \oplus \text{im } G \oplus L^2(M, i\mathbb{R}))$ of the space $L^2(M, E \oplus i\mathbb{R})$. We now define $K_{(\psi, A, t)} : \mathbb{R} \rightarrow L^2(M, E \oplus i\mathbb{R})$ by letting

$$K_{(\psi, A, t)}(x) := x \cdot \frac{d}{dt} \Big|_t \text{SW}_{\eta_t}^{g_t}(\psi, A).$$

Since $\begin{pmatrix} 0 & G \\ G^* & 0 \end{pmatrix}$ yields an isomorphism from $\text{im}(G|_{L_2^2}) \oplus L_1^2(M, i\mathbb{R})$ to $\text{im } G \oplus L^2(M, i\mathbb{R})$, we deduce from surjectivity in (III, 16) that the operator

$$T_{(\psi, A, t)} + K_{(\psi, A, t)} : L_1^2(M, E \oplus i\mathbb{R}) \oplus \mathbb{R} \rightarrow L^2(M, E \oplus i\mathbb{R})$$

is onto. Moreover, its kernel is isomorphic to (III, 16), i.e., to the tangent space of $\widehat{\mathcal{M}}_\eta(\sigma; g)$ at the point (ψ, A, t) . The above considerations show the following: If (ψ, A, t) is an element of the parametrized moduli space, then the tangent space of $\widehat{\mathcal{M}}_\eta(\sigma; g)$ is naturally isomorphic to the 1-dimensional space $\ker(T_{(\psi, A, t)} + K_{(\psi, A, t)})$. In the language of Appendix B, the map K is a stabilizer of T over the parametrized moduli space, hence we have a natural isomorphism of vector bundles

$$T\widehat{\mathcal{M}}_\eta(\sigma; g) = (\det T \rightarrow \widehat{\mathcal{M}}_\eta(\sigma; g)).$$

We now use this observation to show that, given a path $c : [a, b] \rightarrow \widehat{\mathcal{M}}_\eta(\sigma; g)$ which connects two boundary points, the contribution to

$$\text{sw}_{\eta_1}(\sigma; g_1) - \text{sw}_{\eta_{-1}}(\sigma; g_{-1}) \quad (\text{III, 17})$$

given by the endpoints cancel each other out. Summing over all paths yields that (III, 17) vanishes which is the assertion of the theorem.

Writing $c(s) = (\psi(s), A(s), t(s))$, it is immediate from homotopy invariance of the orientation transport that

$$\varepsilon(T_s) = \varepsilon(\psi(a), A(a)) \cdot \varepsilon(T_s^0) \cdot \varepsilon(\psi(b), A(b)), \quad (\text{III, 18})$$

where $T_s := T_{(\psi(s), A(s))}^{g_{t(s)}}$ and

$$T_s^0 := \begin{pmatrix} \mathcal{D}_{A(s)}^{t(s)} & 0 & 0 \\ 0 & -*_{t(s)} d & 2d \\ 0 & 2d^{*t(s)} & 0 \end{pmatrix}.$$

The path T_s^0 is the direct sum of a complex family and the family $\begin{pmatrix} -*_s d & 2d \\ 2d^{*s} & 0 \end{pmatrix}$. As in the proof of Lemma (II, 5.2), part (i), one sees that hence, $\varepsilon(T_s^0) = 1$.

Therefore, it remains to compute $\varepsilon(T_s)$. Since η_1 and η_{-1} are suitable perturbations with respect to g_1 and g_{-1} , the operators T_a and T_b are invertible. Hence, we may apply Lemma (C, 2.3), using the above observation that $K_s := K_{(\psi(s), A(s), t(s))}$ is a stabilizer for T_s . If we choose a parametrization in such a way that $c'(s) \neq 0$, then

$$\ker(T_s + K_s) = \text{Span}_{\mathbb{R}}(\psi'(s), A'(s), 0, t'(s)). \quad (\text{III, 19})$$

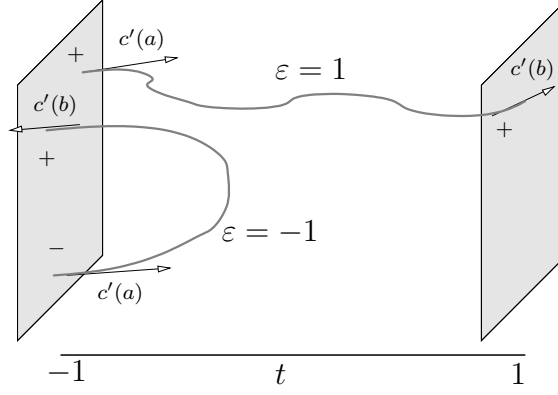


Figure III.2: Computing the orientation transport

Note that the—slightly confusing—ordering of variables stems from the fact that $\ker(T_s + K_s) \subset L^2(M, E \oplus i\mathbb{R}) \oplus \mathbb{R}$, whereas $c'(s) \in L^2(M, E) \oplus \mathbb{R}$. The trivialization (III, 19) of $\ker(T + K)$ induces an isomorphism

$$\Psi_a^b : \ker(T + K)_a \rightarrow \ker(T + K)_b,$$

given by

$$\Psi_a^b(\psi'(a), A'(a), 0, t'(a)) := (\psi'(b), A'(b), 0, t'(b)).$$

The diagram (C, 10) in the case at hand is

$$\begin{array}{ccc} \ker(T_a \oplus K_a) & \xrightarrow{\Psi_a^b} & \ker(T_b \oplus K_b) \\ \downarrow P_{\mathbb{R}} & & \downarrow P_{\mathbb{R}} \\ \mathbb{R} & \xrightarrow{\Phi_{\mathbb{R}}} & \mathbb{R} \end{array}$$

where is uniquely determined by $\Phi_{\mathbb{R}}(t'(a)) = t'(b)$. Observe that it follows from the fact that T_a and T_b are invertible that both, $t'(a)$ and $t'(b)$, are nonzero. According to Lemma (C, 2.3), the orientation transport along T_s is then given by the parity of $\Phi_{\mathbb{R}}$, i.e.,

$$\varepsilon(T_s) = \operatorname{sgn} \frac{t'(b)}{t'(a)}.$$

If c connects elements in distinct parts of the boundary, i.e., $t(b) = -t(a)$, then necessarily $\operatorname{sgn} t'(a) = \operatorname{sgn} t'(b)$ (cf. Fig. III.2). The above formula then implies that $\varepsilon(T_s) = 1$. If on the contrary, c connects elements lying in the same part of the boundary, then the signs of $t'(a)$ and $t'(b)$ differ. Hence, in this case, $\varepsilon(T_s) = -1$. Applying these considerations to (III, 18), we get

$$t(a) \cdot \varepsilon(\psi(a), A(a)) + t(b) \cdot \varepsilon(\psi(b), A(b)) = 0.$$

From this one readily deduces that the contribution given by the endpoints of c to (III, 17) cancel each other out. \square

4 The case $b_1=1$: Wall-crossing formula

If M is a 3-manifold with first Betti number $b_1 = 1$, then the complement of the space of suitable perturbations,

$$\mathcal{P}_k(\sigma) = \{\eta \in Z_k^2(M; i\mathbb{R}) \mid 2\pi ic(\sigma) \neq [\eta]\},$$

is a 1-codimensional affine subspace of $Z_k^2(M; i\mathbb{R})$. The set $Z_k^2(M; i\mathbb{R}) \setminus \mathcal{P}_k$ is called *the wall*, and we denote it by

$$\mathcal{W}_k(\sigma) := \{\eta \in Z_k^2(M; i\mathbb{R}) \mid 2\pi ic(\sigma) = [\eta]\}.$$

Note that

$$\mathcal{W}_k(\sigma) = \eta_0 + d(L_{k+1}^2(M, iT^*M)). \quad (\text{III, 20})$$

The parameter space now decomposes into two connected components which are separated by the wall. In general, we thus cannot choose a path in $\mathcal{P}_k(\sigma)$ connecting arbitrary suitable perturbations. The goal of this section is to establish a so-called *wall-crossing formula* which relates $\text{sw}_{\eta_{-1}}(\sigma)$ to $\text{sw}_{\eta_1}(\sigma)$ if η_{-1} and η_1 lie in different components of $\mathcal{P}_k(\sigma)$.

The original version of the wall-crossing formula seems to be due to Y. Lim [32]. Our presentation largely follows the lecture notes by Nicolaescu [43] and the corresponding proof in the four dimensional case as it can be found in [45], Sec. 2.3.3.

Positive and negative chambers. First of all, we have to provide a way of distinguishing the two components of $\mathcal{P}_k(\sigma)$. This is done via an orientation of the second cohomology of M . Since $b_1 = 1$ such an orientation is given by a closed 2-form μ such that the cohomology class $[\mu]$ in $H^2(M; \mathbb{R})$ is nonzero. If $\eta_0 \in \mathcal{W}_k(\sigma)$, we have $\eta_0 + i\mu \notin \mathcal{W}_k(\sigma)$. This follows from (III, 20) and the fact that the chosen 2-form μ is complementary to $\text{im } d$. This motivates the following definition.

Definition (III, 4.1). Let M be a closed, oriented 3-manifold with $b_1 = 1$. Moreover, let M be equipped with a spin^c structure σ and an orientation of the second cohomology. Let μ be a closed 2-form inducing the given cohomology orientation. Then, for $\eta_0 \in \mathcal{W}_k(\sigma)$, we define the *positive chamber* $\mathcal{P}_k^+(\sigma)$ to be the component of $\mathcal{P}_k(\sigma)$ containing $\eta_0 + i\mu$. The complement

of $\mathcal{P}_k^+(\sigma)$ in $\mathcal{P}_k(\sigma)$ is called the *negative chamber* $\mathcal{P}_k^-(\sigma)$. We thus obtain a decomposition

$$Z_k^2(M; i\mathbb{R}) = \mathcal{P}_k^-(\sigma) \dot{\cup} \mathcal{W}_k(\sigma) \dot{\cup} \mathcal{P}_k^+(\sigma).$$

One readily checks that if μ, μ' and η_0, η'_0 are respectively chosen as above and if $[\mu']$ is a positive multiple of $[\mu]$, then the linear path connecting $\eta_0 + i\mu$ with $\eta'_0 + i\mu'$ never crosses the wall. Therefore, the above definition depends only on the chosen orientation of $H^2(M; \mathbb{R})$. Moreover, note that if μ is harmonic with respect to some Riemannian metric, then

$$\mathcal{P}_k^\pm(\sigma) = \eta_0 + \{ \eta \in Z_k^2(M, i\mathbb{R}) \mid \pm(\eta, i\mu)_{L^2} > 0 \}$$

and

$$\mathcal{W}_k(\sigma) = \eta_0 + \{ \eta \in Z_k^2(M, i\mathbb{R}) \mid (\eta, i\mu)_{L^2} = 0 \}.$$

Exactly as in Theorem (III, 3.2) one concludes that the number $\text{sw}_\eta(\sigma; g)$ is independent of η and g as long as η is a suitable perturbation taken from only one of the two chambers. We thus have the following result:

Proposition (III, 4.2). *Let (M, σ) be a closed, oriented spin^c 3-manifold with $b_1 = 1$. Moreover, let M be equipped with an orientation of the second cohomology. Then, for an arbitrary Riemannian metric g and corresponding suitable perturbations $\eta^\pm \in \mathcal{P}_k^\pm(\sigma)$ the numbers*

$$\text{sw}^\pm(\sigma) := \text{sw}_{\eta^\pm}(\sigma; g)$$

are independent of g and η^\pm .

The circle of reducibles. Whenever $\eta_0 \in \mathcal{W}_k(\sigma)$, we know from Proposition (III, 2.7) that the reducible part of $\mathcal{M}_{\eta_0}(\sigma)$ is homeomorphic to the circle $H_{dR}^1(M; i\mathbb{R})/H_{dR}^1(M; 4\pi i\mathbb{Z})$. If we let ω_0 be a generator of the lattice $H_{dR}^1(M; 4\pi i\mathbb{Z})$, then it follows from the proof of Proposition (III, 2.7) that the circle of reducibles is parametrized by $\{A_0 + r\omega_0\}_{r \in [0,1]}$, where A_0 is a fixed reducible η_0 -monopole.

If M is equipped with a Riemannian metric g , we may assume that ω_0 is harmonic with respect to g . This implies $*\omega_0 \in i\mathcal{H}^2(M; g)$ so that $*\omega_0$ defines an orientation of $i\mathcal{H}^2(M; g)$ and thus an orientation of the second cohomology.

Definition (III, 4.3). Let (M, g) be a closed, oriented Riemannian 3-manifold with $b_1 = 1$, equipped with an orientation of $H^2(M; \mathbb{R})$. Moreover, let σ be a spin^c structure on M , and let $\eta_0 \in \mathcal{W}_k(\sigma)$. A harmonic 1-form $\omega_0 \in i\mathcal{H}^1(M; g)$ is called a *generator of reducible η_0 -monopoles* if $[\omega_0]$ generates the lattice $H_{dR}^1(M; 4\pi i\mathbb{Z})$ and if $*\omega_0$ induces the given orientation of the second cohomology.

Note that this characterizes ω_0 uniquely since there are only two possible generators of $H_{dR}^1(M; 4\pi i\mathbb{Z})$.

Finding a suitable path. To find the relation between $\text{sw}^+(\sigma)$ and $\text{sw}^-(\sigma)$ we have to consider a path connecting two suitable perturbations $\eta^\pm \in \mathcal{P}^\pm(\sigma)$. We will then derive the wall-crossing formula from a detailed analysis of the parametrized moduli space near the circle of reducibles. Finding an appropriate path requires some preliminary considerations.

Proposition (III, 4.4). *Let (M, g) be a closed, oriented Riemannian 3-manifold with $b_1 = 1$. Moreover, let M be equipped with an orientation of the second cohomology and a spin^c structure σ . For $\eta_0 \in \mathcal{W}_k(\sigma)$, let ω_0 be the generator of reducible η_0 -monopoles. Then for a generic element A in $\mathcal{A}(\sigma)$, the family $\{\mathcal{D}_{A+r\omega_0}\}_{r \in [0,1]}$ is transversal with only simple crossings, cf. Definition (C, 1.7).*

Proof. We use the perturbation methods of Section 1. Consider the open set $X := \{\psi \in L_1^2(M, S) \mid \psi \neq 0\}$ of non-vanishing spinors, and define a vector bundle $V \rightarrow X$ via

$$V_\psi := \ker \text{Re}(i\psi, \cdot)_{L^2}, \quad 0 \neq \psi \in L_1^2(M, S).$$

Then $V \rightarrow X$ is a bundle of \mathbb{R} -Hilbert spaces. More precisely, V_ψ is a subspace of $L_1^2(M, S)$, which we consider as an \mathbb{R} -Hilbert space. V_ψ has codimension 1 because

$$(\ker \text{Re}(i\psi, \cdot)_{L^2})^\perp = \text{Span}_{\mathbb{R}} i\psi.$$

Next consider the section $\Phi : X \rightarrow V$ given by

$$\Phi(\psi) := \mathcal{D}_{A_0}\psi,$$

where A_0 is a fixed gauge field. Observe that Φ is well-defined since formal self-adjointness of the Dirac operator implies that $\text{Re}(i\psi, \mathcal{D}_{A_0}\psi)_{L^2} = 0$. At a zero ψ of Φ we have

$$D_\psi \Phi = \mathcal{D}_{A_0} : L_1^2(M, S) \rightarrow \ker \text{Re}(i\psi, \cdot)_{L^2}.$$

Hence, $D_\psi \Phi$ is a Fredholm operator of index 1. This is because the Fredholm operator $\mathcal{D}_{A_0} : L_1^2(M, S) \rightarrow L^2(M, S)$ of index 0 produces a Fredholm operator index 1 when the target space is restricted to a 1-codimensional subspace.

Proceeding as in Section 1, we now perturb Φ to make it transversal to the zero section. As the perturbation space P we take the space of imaginary

valued 1-forms of Sobolev class 1, and the perturbation map is defined as the section $\hat{\Phi}$ of $\text{pr}_1^* V \rightarrow X \times P$ given by

$$\hat{\Phi}(\psi, a) := \mathcal{D}_{A_0+a}\psi.$$

The next thing to establish is that for each zero (ψ, a) of $\hat{\Phi}$, the differential $D_{(\psi,a)}\hat{\Phi}$ is surjective. We thus compute

$$D_{(\psi,a)}\hat{\Phi}(\varphi, b) = \mathcal{D}_{A_0+a}\varphi + \frac{1}{2}c(b)\psi. \quad (\text{III}, 21)$$

Assume that φ_0 is orthogonal to the closed subspace $\text{im } D_{(\psi,a)}\hat{\Phi}$. Then the above equation shows that for every $\varphi \in L_1^2(M, S)$, the scalar product $\text{Re}(\mathcal{D}_{A_0+a}\varphi, \varphi_0)_{L^2}$ vanishes. In combination with formal self-adjointness of the Dirac operator, this yields that

$$\varphi_0 \in \ker \mathcal{D}_{A_0+a}.$$

On the other hand, according to equation (III, 21), the scalar product $\text{Re}(c(b)\psi, \varphi_0)_{L^2}$ vanishes for all 1-forms b . According to part (i) of Proposition (I, 1.2), we thus have

$$(b, q(\psi, \varphi_0))_{L^2} = 0$$

for each 1-form b . This implies that $q(\psi, \varphi_0) = 0$. As in the proof of Proposition (III, 2.3) we may therefore deduce that there exists $f \in L_1^2(M, i\mathbb{R})$ such that

$$\varphi_0 = f \cdot \psi.$$

From the fact that $\mathcal{D}_{A_0+a}\varphi_0 = 0$ we infer that

$$0 = \mathcal{D}_{A_0+a}(f \cdot \psi) = f \cdot \mathcal{D}_{A_0+a}\psi + c(df)\psi = c(df)\psi.$$

This shows that $df = 0$, since $\psi \neq 0$ and can thus only vanish on the complement of a dense open set. Therefore, f is an imaginary constant and φ_0 is a multiple of ψ by f . As $\varphi_0 \in \ker \text{Re}(i\psi, \cdot)_{L^2}$, this demands $\varphi_0 \equiv 0$. Hence, $D_{(\psi,a)}\hat{\Phi}$ is surjective.

We now apply similar considerations as in Proposition (III, 1.4) and find that

$$\hat{\Psi} : X \times [0, 1] \times P \rightarrow \text{pr}_1^* V, \quad \hat{\Psi}(\psi, r, a) = \mathcal{D}_{A+a+r\omega_0}\psi,$$

is transversal to the zero section. Therefore, given a generic $a \in P$, the set

$$\bigcup_{r \in [0,1]} (\ker \mathcal{D}_{A_0+a+r\omega_0} \setminus \{0\}) \times \{r\} \quad (\text{III}, 22)$$

is either empty or a 2-dimensional real submanifold of $X \times [0, 1]$. Let us assume the latter holds (otherwise, there is nothing left to prove). Then the projection

$$\bigcup_{r \in [0, 1]} (\ker \mathcal{D}_{A_0+a+r\omega_0} \setminus \{0\}) \times \{r\} \rightarrow [0, 1]$$

is a smooth map from a 2-dimensional manifold to a 1-dimensional one. By virtue of (the finite dimensional version of) Sard's Theorem, we can ascertain that for each r chosen from a dense subset of $[0, 1]$, the set

$$\ker (\mathcal{D}_{A_0+a+r\omega_0}) \setminus \{0\}$$

is either empty or a manifold of \mathbb{R} -dimension 1. On the other hand, if nonempty, it necessarily has \mathbb{C} -dimension ≥ 1 . This can only be possible if this manifold is empty and therefore, $\ker \mathcal{D}_{A_0+a+r\omega_0} = 0$ for every r in a dense subset of $[0, 1]$. For any other r , the kernel of $\mathcal{D}_{A_0+a+r\omega_0}$ is of \mathbb{R} -dimension 2, since otherwise (III, 22) could not form a 2-dimensional manifold.

The last thing remaining to prove is that the family $\{\mathcal{D}_{A_0+a+r\omega_0}\}_{r \in [0, 1]}$ is transversal if a is chosen from the generic subset. Fix $r_0 \in [0, 1]$ such that the corresponding operator $\mathcal{D}_{A_0+a+r_0\omega_0}$ is not invertible, and let ψ_0 be an element of the kernel, of norm 1. Transversality of the section $(\psi, r) \mapsto \hat{\Psi}(\psi, r, a)$ guarantees that

$$D_{(\psi_0, r_0)} \hat{\Psi}(\varphi, r, a) = \mathcal{D}_{A_0+a+r_0\omega_0} \varphi + r \cdot \frac{d}{ds} \Big|_{r=r_0} \mathcal{D}_{A_0+a+r\omega_0} \psi_0$$

yields a surjection $L_1^2(M, S) \oplus \mathbb{R} \rightarrow \ker \operatorname{Re}(i\psi_0, \cdot)_{L^2}$. In particular, there exists $(\varphi, r) \in L_1^2(M, S) \oplus \mathbb{R}$ such that

$$\mathcal{D}_{A_0+a+r_0\omega_0} \varphi + r \cdot \frac{d}{ds} \Big|_{r=r_0} \mathcal{D}_{A_0+a+r\omega_0} \psi_0 = \psi_0.$$

Since ψ_0 is a nonzero harmonic spinor with respect to $\mathcal{D}_{A_0+a+r_0\omega_0}$, taking the L^2 -product with ψ_0 enforces that

$$\left(\frac{d}{ds} \Big|_{r=r_0} \mathcal{D}_{A_0+a+r\omega_0} \psi_0, \psi_0 \right)_{L^2} \neq 0.$$

This shows that the zero eigenvalue at the point r_0 crosses transversally. \square

Corollary (III, 4.5). *Let (M, g) be a closed, oriented Riemannian 3-manifold with $b_1 = 1$. Moreover, let M be equipped with an orientation of the second cohomology and a spin^c structure σ . Suppose that $\eta_{\pm 1} \in \mathcal{P}_k^\pm(\sigma)$ are suitable perturbations with respect to g . Then there exists a connecting C^3 -path*

$$\eta : [-1, 1] \rightarrow Z_k^2(M; i\mathbb{R})$$

such that

- (i) *The path η meets the wall transversally and does so only in 0. That is, $\eta_t \in \mathcal{W}_k(\sigma)$ if and only if $t = 0$ and $(\eta'_0, *\omega_0)_{L^2} > 0$, where ω_0 is the generator of reducible η_0 -monopoles.*
- (ii) *The irreducible part of the parametrized moduli space is either empty or a 1-dimensional C^3 -submanifold of $\mathcal{B}^* \times [-1, 1]$.*
- (iii) *If A_0 is a reducible η_0 -monopole, then the family $\{\mathcal{D}_{A_0+r\omega_0}\}_{r \in [0,1]}$ is transversal with only simple crossings.*

Proof. Fix a gauge field A_0 lying in the generic set given by Proposition (III, 4.4). We define $\eta_0 := -F_{A_0} \in Z_k^2(M; i\mathbb{R})$, forcing A_0 to become a reducible η_0 -monopole. In particular, $\eta_0 \in \mathcal{W}_k(\sigma)$. We now provide an appropriate perturbation space to employ the results of Section (III, 1.1) again. Let us additionally fix a constant $C > 0$ and consider the set of all C^3 -paths $\eta : [-1, 1] \rightarrow Z_k^2(M; i\mathbb{R})$ satisfying

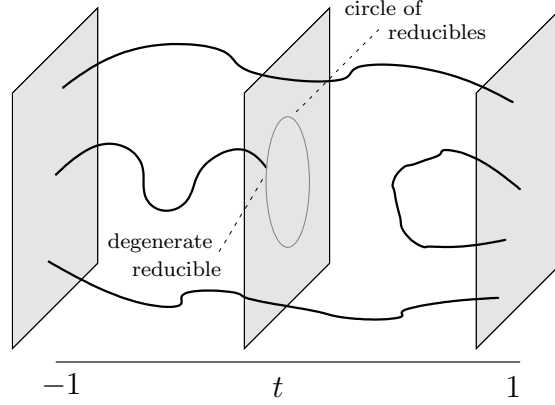
- $\eta(t) \in \mathcal{P}_k^\pm(\sigma)$ if $\pm t > 0$,
- $\eta(i) = \eta_i$ for $i \in \{-1, 0, 1\}$,
- $(\eta'(0), *\omega_0)_{L^2} > C$,

where ω_0 is the generator of reducible η_0 -monopoles. Equipped with the C^3 -topology, this set becomes a Banach manifold since it is an open subset of the affine Banach space

$$\{\eta : [-1, 1] \rightarrow Z_k^2(M; i\mathbb{R}) \mid \eta \text{ is } C^3 \text{ and } \eta(i) = \eta_i \text{ for } i \in \{-1, 0, 1\}\}.$$

Note for this, that the requirement $(\eta'(0), *\omega_0)_{L^2} > C$ is an open condition. It implies that near 0 a path η satisfies $\eta(t) \in \mathcal{P}_k^\pm(\sigma)$ if $\pm t > 0$. Away from 0, the first property is also an open condition since $\mathcal{P}_k^\pm(\sigma)$ are open subsets of $Z_k^2(M; i\mathbb{R})$.

We now use the above Banach manifold as a perturbation space. Each corresponding path automatically satisfies (i) and (iii). Similarly as in the proof of Theorem (III, 3.2), an application of Proposition (III, 1.1) and the Sard-Smale Theorem yields that for a generic choice of such η , property (ii) can also be achieved. \square

Figure III.3: The parametrized moduli space in the case $b_1 = 1$

The singular cobordism. Let us now fix a path η as in the above Corollary. As we have seen in Section 3, the parametrized moduli space

$$\widehat{\mathcal{M}}_\eta(\sigma) = \bigcup_{t \in [-1, 1]} \mathcal{M}_{\eta_t}(\sigma) \times \{t\}$$

is compact. However, it does in general not necessarily form a C^3 -cobordism between the moduli spaces $\mathcal{M}_{\eta_{-1}}(\sigma; g)$ and $\mathcal{M}_{\eta_1}(\sigma; g)$ for singularities may occur at $\mathcal{M}_{\eta_0}(\sigma; g) \times \{0\}$, i.e., when the path η crosses the wall. As the reducible part of $\mathcal{M}_{\eta_0}(\sigma)$ is a circle, the singular cobordism will look roughly as in Fig. III.3.

We now have to understand the nature of these singularities. Due to property (ii) of the path η , they may only occur at the circle of reducibles.

Definition (III, 4.6). Let (M, g) be a closed, oriented Riemannian 3-manifold. Moreover, let σ be a spin^c structure on M and let $\eta_0 \in \mathcal{W}_k(\sigma)$. Then a reducible η_0 -monopole A is called *non-degenerate* if \mathcal{D}_A is invertible and *slightly degenerate* provided that $\dim_{\mathbb{C}} \ker \mathcal{D}_A = 1$.

In the case at hand, since the reducible part of $\mathcal{M}_{\eta_0}(\sigma)$ can be parametrized by $\{A_0 + r\omega_0\}_{r \in [0, 1]}$, part (iii) of Corollary (III, 4.5) ensures that there may only occur reducibles which are at most slightly degenerate.

Local structure near a reducible. The difficulty in understanding the parametrized moduli space near the circle of reducibles stems from the fact that reducible points lie in a different stratum of the quotient \mathcal{B} than the irreducible part of the moduli space. Recall that according to the slice theorem, the local model is as follows: If $(0, A_0) \in \mathcal{C}$ is a reducible configuration,

then there exists a U_1 -invariant open neighbourhood U of 0 in the slice, i.e.,

$$U \subset \ker(G_{(0,A)}^*|_{L_1^2}) = L_1^2(M, S) \oplus \ker(d^*|_{L_1^2}),$$

such that U/U_1 models an open neighbourhood of $[0, A_0]$ in \mathcal{B} . Recall that in accordance with our earlier considerations, U_1 acts only on the spinor part of $L_1^2(M, S) \oplus \ker(d^*|_{L_1^2})$.

If $\eta : [-1, 1] \rightarrow Z_k^2(M, i\mathbb{R})$ is a path of perturbations such that η_0 admits a reducible monopole $(0, A_0)$, then the above shows that the parametrized moduli space $\widehat{\mathcal{M}}_\eta(\sigma)$ near $[0, A_0, 0]$ is homeomorphic to

$$\{(\varphi, a, t) \in U \times (-\varepsilon, \varepsilon) \mid \text{SW}_{\eta_t}(\varphi, A_0 + a) = 0\} / U_1, \quad (\text{III, 23})$$

where U is a suitable U_1 -invariant open neighbourhood of 0 in the slice at $(0, A_0)$ given by the slice theorem. To understand the above zero set, we let

$$s : L_1^2(M, S) \oplus \ker(d^*|_{L_1^2}) \times (-\varepsilon, \varepsilon) \longrightarrow L^2(M, S) \oplus \ker d^* \quad (\text{III, 24})$$

be defined by

$$\begin{aligned} s(\varphi, a, t) &:= \text{Proj}_{L^2(M, S) \oplus \ker d^*} \circ \text{SW}_{\eta_t}(\varphi, A_0 + a) \\ &= \text{Proj}_{L^2(M, S) \oplus \ker d^*} \left(\mathcal{D}_{A_0+a} \varphi, \frac{1}{2} q(\varphi) - *(F_{A_0+a} + \eta_t) \right). \end{aligned}$$

Therefore, $s(\varphi, a, t)$ and $\text{SW}_{\eta_t}(\varphi, A_0 + a)$ coincide whenever the 1-form factor of $\text{SW}_{\eta_t}(\varphi, A_0 + a)$ is co-closed. Since $s(\varphi, a, t) = 0$ implies that φ is harmonic, Proposition (I, 2.1) shows that this is true at every zero of s , i.e.,

$$s(\varphi, a, t) = 0 \iff \text{SW}_{\eta_t}(\varphi, A_0 + a) = 0.$$

Hence, the local model (III, 23) can be replaced by a neighbourhood of 0 in $s^{-1}(0)/U_1$. Summarizing this, we have the following:

Lemma (III, 4.7). *Let (M, g) be a closed, oriented Riemannian 3-manifold, and let $\eta : [-1, 1] \rightarrow Z_k^2(M, i\mathbb{R})$ be a path of perturbations such that η_0 admits a reducible monopole $(0, A_0)$. Then there exists a U_1 -invariant neighbourhood U of 0 in $L_1^2(M, S) \oplus \ker(d^*|_{L_1^2})$ such that*

$$\{(\varphi, a, t) \in U \times (-\varepsilon, \varepsilon) \mid s(\varphi, a, t) = 0\} / U_1$$

is homeomorphic to an open neighbourhood of $[0, A_0, 0]$ in $\widehat{\mathcal{M}}_\eta(\sigma)$.

For the remainder of this section, we consider the situation of the last paragraph, i.e., we shall always assume that $b_1 = 1$ and that η is chosen as in Corollary (III, 4.5). Moreover, we fix a reducible η_0 -monopole $(0, A_0)$, and

let ω_0 denote the generator of reducible η_0 -monopoles. To study $\widehat{\mathcal{M}}_\eta(\sigma)$ near $[0, A_0, 0]$, we compute that the differential of s at 0,

$$D_0s : L_1^2(M, S) \oplus \ker(d^*|_{L_1^2}) \oplus \mathbb{R} \rightarrow L^2(M, S) \oplus \ker d^*,$$

is given by

$$\begin{aligned} D_0s(\varphi, a, t) &= \text{Proj}_{L^2(M, S) \oplus \ker d^*} (\mathcal{D}_{A_0}\varphi, - * da - *(D_0\eta)(t)) \\ &= (\mathcal{D}_{A_0}\varphi, - * da - t * \eta'_0), \end{aligned}$$

where we are using that $*da$ and $*\eta'_0$ are co-closed. Observe that $da + t \cdot \eta'_0 = 0$ if and only if $da = 0$ and $t = 0$. This is because $(\eta'_0, *\omega_0)_{L^2} > 0$ so that necessarily $[\eta'_0] \neq 0$, whereas $[da] = 0$. We conclude that

$$\begin{aligned} \ker D_0s &= \ker \mathcal{D}_{A_0} \oplus \ker d|_{\ker d^*} \oplus \{0\} \\ &= \ker \mathcal{D}_{A_0} \oplus i\mathcal{H}^1(M) \oplus \{0\}. \end{aligned}$$

The cokernel of D_0s is given by the orthogonal complement of $\text{im } D_0s$ in $L^2(M, S) \oplus \ker d^*$. Since $\ker d^*$ is spanned by η'_0 and the image of $*d$, we deduce that

$$\text{coker } D_0s = \text{coker } \mathcal{D}_{A_0} \oplus \{0\} = \ker \mathcal{D}_{A_0} \oplus \{0\}.$$

Lemma (III, 4.8). *If A_0 is non-degenerate, then a neighbourhood of $[0, A_0, 0]$ in the parametrized moduli space $\widehat{\mathcal{M}}_\eta(\sigma)$ is homeomorphic to a neighbourhood of A_0 in the circle of reducibles.*

Proof. Recall that the circle of reducibles near $[0, A_0, 0]$ is given by

$$\{[0, A_0 + r\omega_0, 0] \in \mathcal{B} \times [-1, 1] \mid r \in (-\delta, \delta)\},$$

where $\delta < \frac{1}{2}$, and ω_0 is the generator of reducible η_0 -monopoles. In terms of the local model $s^{-1}(0)/U_1$, this corresponds to the line

$$\{(0, r\omega_0, 0) \in L_1^2(M, S) \oplus \ker d^*|_{L_1^2} \times (-\varepsilon, \varepsilon) \mid r \in (-\delta, \delta)\}, \quad (\text{III, 25})$$

where we are using that U_1 acts only on the spinor part, hence has no effect on the above set. As $\ker \mathcal{D}_{A_0} = 0$ at a non-degenerate monopole, the differential D_0s is surjective and has kernel equal to $i\mathcal{H}^1(M)$. As this is 1-dimensional, the implicit function theorem implies that (III, 25) is exactly the zero set of s near 0. \square

Remark. The fact that $\ker D_0s = i\mathcal{H}^1(M)$ motivates the remark following (II, 22) that $i\mathcal{H}^1(M)$ is the tangent space to the reducible part of the moduli space.

Proposition (III, 4.9). *Let A_0 be slightly degenerate and fix $\psi_0 \in L_1^2(M, S)$ of norm 1 spanning $\ker \mathcal{D}_{A_0}$. Then in a neighbourhood of 0 the following holds:*

$$s(\varphi, a, t) = 0 \iff (\varphi, a, t) = \begin{cases} (0, r\omega_0, 0) & \text{if } \varphi = 0 \\ (z\psi_0, g(z)\omega_0, 0) + f(z, g(z)) & \text{if } \varphi \neq 0 \end{cases},$$

with small $(z, r) \in \mathbb{C} \times \mathbb{R}$, and where ω_0 is the generator of reducible η_0 -monopoles. The map f is a U_1 -equivariant C^3 -map

$$f : \mathbb{C} \times \mathbb{R} \rightarrow (\ker \mathcal{D}_{A_0} \oplus i\mathcal{H}^1)^\perp,$$

where the orthogonal complement is taken in $L_1^2(M, S) \oplus \ker(d^*|_{L_1^2}) \oplus \mathbb{R}$. The map $g : \mathbb{C} \rightarrow \mathbb{R}$ is C^1 and U_1 -invariant. Both maps vanish of second order in 0.

Proof. We will use the so-called local Kuranishi technique to study the zero set of s near 0. Let

$$\Phi := \text{Proj}_{\text{im } D_0 s} \circ s \quad \text{and} \quad \Psi := \text{Proj}_{\text{coker } D_0 s} \circ s.$$

As $s = \Phi \oplus \Psi$, we have to find the common zeros of Φ and Ψ . Since we have forced Φ to have a surjective differential at 0, we can apply the implicit function theorem to obtain a C^3 -map

$$f : \ker \mathcal{D}_{A_0} \oplus i\mathcal{H}^1 \rightarrow (\ker \mathcal{D}_{A_0} \oplus i\mathcal{H}^1)^\perp$$

such that the graph of f locally describes the zero set of Φ . More precisely, using coordinates (z, r) with respect to (ψ_0, ω_0) on $\ker \mathcal{D}_{A_0} \oplus i\mathcal{H}^1$, we have for small (φ, a, t) that

$$\Phi(\varphi, a, t) = 0 \iff (\varphi, a, t) = (z\psi_0, r\omega_0, 0) + f(z, r).$$

The first observation is that $f(0, r) = 0$ for every r . This follows from the fact that $\Phi(0, r\omega_0, 0) = 0$. Moreover, as the zero set of Φ is U_1 -invariant, we infer that f is necessarily U_1 -equivariant, i.e.,

$$f(\lambda z, r) = \lambda f(z, r), \quad \lambda \in U_1.$$

Recall that on the right hand side, λ only operates on the spinor part.

To extract information about the zeros of s we now investigate

$$\hat{\Psi} : \mathbb{C} \times \mathbb{R} \rightarrow \ker \mathcal{D}_{A_0}, \quad \hat{\Psi}(z, r) := \Psi((z\psi_0, r\omega_0, 0) + f(z, r)).$$

This map is called the *Kuranishi obstruction map*. It reduces the original infinite dimensional problem of finding the zeros of s to finite dimensions. From the corresponding property of f it is immediate that $\hat{\Psi}(0, r) = 0$ for every r and that $\hat{\Psi}$ is a U_1 -equivariant map. Note that the latter implies that for fixed r , the map $\hat{\Psi}(\cdot, r)$ is complex differentiable in 0. Therefore, we can factor out z writing

$$\hat{\Psi}(z, r) = z \cdot \hat{\Psi}_1(z, r), \quad \text{where } \hat{\Psi}_1(z, r) := \begin{cases} \frac{\hat{\Psi}(z, r)}{z}, & z \neq 0, \\ \frac{\partial}{\partial z} \Big|_{(0, r)} \hat{\Psi}(z, r), & z = 0. \end{cases}$$

Since $\hat{\Psi}(z, r)$ is C^3 , the map $\hat{\Psi}_1$ is at least C^1 . This easily follows from the Taylor Formula. We now claim that

$$\frac{\partial}{\partial r} \Big|_{(0, 0)} \hat{\Psi}_1(z, r) = \kappa \cdot \psi_0, \quad \text{where } \kappa := \frac{1}{2}(c(\omega_0)\psi_0, \psi_0)_{L^2}. \quad (\text{III, 26})$$

Observe that $\frac{\partial}{\partial r} \Big|_{(0, 0)} \hat{\Psi}_1(z, r) = \frac{\partial^2}{\partial r \partial z} \Big|_{(0, 0)} \hat{\Psi}(z, r)$. Making use of the definition of Ψ and letting $\Pi := \text{Proj}_{\ker \mathcal{D}_{A_0}}$ we have

$$\begin{aligned} \frac{\partial}{\partial z} \Big|_{(0, r)} \hat{\Psi}(z, r) &= \frac{\partial}{\partial z} \Big|_{(0, r)} \Pi \circ s \circ ((z\psi_0, r\omega_0, 0) + f(z, r)) \\ &= \Pi \circ D_{(0, r\omega_0, 0)} s \circ \left(\frac{\partial}{\partial z} \Big|_{(0, r)} (z\psi_0, r\omega_0, 0) + \frac{\partial}{\partial z} \Big|_{(0, r)} f(z, r) \right) \\ &= \Pi \circ (\mathcal{D}_{A_0 + r\omega_0} \psi_0) + \Pi \circ \frac{\partial}{\partial z} \Big|_{(0, r)} (\mathcal{D}_{A_0 + a(z, r)} \varphi(z, r)), \end{aligned}$$

where $\varphi(z, r)$ and $a(z, r)$ denote the spinor and the 1-form part of $f(z, r)$ respectively. Note that we have written down only the spinor part of $D_{(0, r\omega_0, 0)} s$ which is sufficient since the 1-form part vanishes when $\Pi = \text{Proj}_{\ker \mathcal{D}_{A_0}}$ is applied. Since $\psi_0 \in \ker \mathcal{D}_{A_0}$, we find that

$$\Pi \circ (\mathcal{D}_{A_0 + r\omega_0} \psi_0) = (\mathcal{D}_{A_0 + r\omega_0} \psi_0, \psi_0)_{L^2} = \frac{1}{2}(c(r\omega_0)\psi_0, \psi_0)_{L^2} = \kappa \cdot r.$$

The second term in the above expression of $\frac{\partial}{\partial z} \Big|_{(0, r)} \hat{\Psi}(z, r)$ is equal to zero because

$$\begin{aligned} \Pi \circ \frac{\partial}{\partial z} \Big|_{(0, r)} (\mathcal{D}_{A_0 + a(z, r)} \varphi(z, r)) &= \frac{\partial}{\partial z} \Big|_{(0, r)} (\mathcal{D}_{A_0 + a(z, r)} \varphi(z, r), \psi_0)_{L^2} \\ &= \frac{\partial}{\partial z} \Big|_{(0, r)} (\mathcal{D}_{A_0} \varphi(z, r), \psi_0)_{L^2} + \frac{1}{2} \frac{\partial}{\partial z} \Big|_{(0, r)} (c(a(z, r))\varphi(z, r), \psi_0)_{L^2} \\ &= \frac{\partial}{\partial z} \Big|_{(0, r)} (\mathcal{D}_{A_0} \varphi(z, r), \psi_0)_{L^2} + \frac{1}{2} \left(c \left(\frac{\partial}{\partial z} \Big|_{(0, r)} a(z, r) \right) \varphi(0, r) \right. \\ &\quad \left. + c(a(0, r)) \left(\frac{\partial}{\partial z} \Big|_{(0, r)} \varphi(z, r) \right), \psi_0 \right)_{L^2}. \end{aligned}$$

Here, the first summand vanishes as ψ_0 is harmonic and \mathcal{D}_{A_0} is formally self-adjoint, while the second and the third term equal zero since $a(0, r) = \varphi(0, r) = 0$. For this recall that $f(0, r) = 0$. Putting these computations together proves (III, 26). Note that we have also proved that $\hat{\Psi}_1(0, 0) = 0$.

The next observation is that

$$\operatorname{sgn} \kappa = \operatorname{SF}(\mathcal{D}_{A_0+r\omega_0}; |r| \ll 1). \quad (\text{III, 27})$$

This is because the family $\{\mathcal{D}_{A_0+r\omega_0}\}$ has only simple crossings so that

$$\operatorname{SF}(\mathcal{D}_{A_0+r\omega_0}; |r| \ll 1) = \operatorname{sgn} \left(\frac{d}{dr} \Big|_{r=0} \mathcal{D}_{A_0+r\omega_0} \psi_0, \psi_0 \right)_{L^2} = \operatorname{sgn} \frac{1}{2} (c(\omega_0) \psi_0, \psi_0)_{L^2}.$$

In particular, $\kappa \neq 0$ since the family is transversal.

Combining this information with (III, 26) shows that $\hat{\Psi}_1(0, 0) = 0$ and $\frac{\partial}{\partial r} \Big|_{(0,0)} \hat{\Psi}_1(z, r) \neq 0$. This allows us to apply the implicit function theorem to the map $\hat{\Psi}_1$ near 0 which produces a C^1 -map $g : \mathbb{C} \rightarrow \mathbb{R}$ such that in a neighbourhood of 0,

$$\hat{\Psi}_1(z, r) = 0 \iff r = g(z).$$

By definition of $\hat{\Psi}$ and according to the equivariance property of f , one readily ensures that the map g must be U_1 -invariant.

Putting all pieces of information together yields

$$\begin{aligned} s(\varphi, a, t) = 0 &\iff \Phi(\varphi, a, t) = 0 \text{ and } \Psi(\varphi, a, t) = 0 \\ &\iff (\varphi, a, t) = (z\psi_0, r\omega_0, 0) + f(z, r) \text{ and } \hat{\Psi}(z, r) = 0 \\ &\iff (\varphi, a, t) = \begin{cases} (0, r\omega_0, 0) & \text{if } \varphi = 0 \\ (z\psi_0, g(z)\omega_0, 0) + f(z, g(z)) & \text{if } \varphi \neq 0. \end{cases} \end{aligned}$$

From the fact that g is a U_1 -invariant map satisfying $g(0) = 0$, we immediately obtain that it vanishes of second order. Moreover, if we write

$$f(z, g(z)) = (\varphi(z), a(z), t(z)),$$

then U_1 -equivariance of f means

$$\varphi(\lambda z) = \lambda \varphi(z), \quad a(\lambda z) = a(z) \quad \text{and} \quad t(\lambda z) = t(z), \quad \lambda \in U_1.$$

Hence, a and t are also U_1 -invariant and thus vanish of second order in 0. To compute $\varphi'(0)$ we note that

$$\begin{aligned} 0 &= D_0 s \left(\frac{\partial}{\partial z} \Big|_{z=0} (z\psi_0, g(z)\omega_0, 0) + \frac{\partial}{\partial z} \Big|_{z=0} f(z, g(z)) \right) \\ &= (\mathcal{D}_{A_0} \psi_0 + \mathcal{D}_{A_0} \varphi'(0), 0) = (\mathcal{D}_{A_0} \varphi'(0), 0), \end{aligned}$$

where we are using that $a'(0) = g'(0) = 0$ and that ψ_0 is harmonic with respect to \mathcal{D}_{A_0} . We deduce that $\mathcal{D}_{A_0}\varphi'(0) = 0$. On the other hand, $\varphi'(0) \perp \ker \mathcal{D}_{A_0}$ for the image of f is contained in $(\ker \mathcal{D}_{A_0} \oplus i\mathcal{H}^1)^\perp$. Hence, $\varphi'(0) = 0$ which shows that φ also has a zero of order 2 in 0. \square

Remark. From the U_1 -equivariance and U_1 -invariance properties and the fact that f and g vanish of second order in 0 one deduces from the above proposition that a neighbourhood of $[0, A_0, 0]$ in $\widehat{\mathcal{M}}_\eta(\sigma)$ is homeomorphic to a neighbourhood of 0 in

$$\{(z, x) \in \mathbb{R}_+^0 \times \mathbb{R} \mid z = 0 \text{ or } x = 0\},$$

where $\mathbb{R}_+^0 := \mathbb{R}_+ \cup \{0\}$. The branch $\mathbb{R}_+ \times \{0\}$ corresponds to the irreducible part of the parametrized moduli space near $[0, A_0, 0]$.

The next result shows that the spectral flow of $\{\mathcal{D}_{A_0+r\omega_0}\}$ at $r = 0$ determines whether the irreducible branch hits the circle of reducibles coming from the left or from the right. More precisely,

Proposition (III, 4.10). *If A_0 is a slightly degenerate η_0 -monopole, then the following holds:*

- *If $\text{SF}(\mathcal{D}_{A_0+r\omega_0}; |r| \ll 1) = -1$, then the irreducible part of $\widehat{\mathcal{M}}_\eta(\sigma)$ near $[0, A_0, 0]$ is entirely contained in $\mathcal{B}^* \times [-1, 0]$.*
- *If $\text{SF}(\mathcal{D}_{A_0+r\omega_0}; |r| \ll 1) = 1$, then the irreducible branch lies in $\mathcal{B}^* \times (0, 1]$.*

Moreover, if $[\psi, A, t]$ is an element of the irreducible branch close³ to $[0, A_0, 0]$, then

$$\varepsilon(\psi, A) = 1.$$

Proof. As the path η is chosen according to Corollary (III, 4.5), we infer from part (i) of this result that

$$\eta_t \in \mathcal{P}_k^\pm(\sigma) \iff \pm t > 0.$$

Let f and g be as in Proposition (III, 4.9), and write $f(z, g(z)) = (\varphi(z), a(z), t(z))$. Then the first part of the proposition will be established if we prove that that

$$\text{SF}(\mathcal{D}_{A_0+r\omega_0}; |r| \ll 1) = \pm 1 \iff \eta_{t(z)} \in \mathcal{P}_k^\pm(\sigma) \quad \text{for } |z| \ll 1.$$

By definition of ω_0 , this amounts to the same as proving that

$$\text{SF}(\mathcal{D}_{A_0+r\omega_0}; |r| \ll 1) = \text{sgn}(\eta_{t(z)} - \eta_0, *\omega_0)_{L^2} \quad \text{for } |z| \ll 1. \quad (\text{III, 28})$$

³The proof will provide a more precise meaning for that.

We recall that $(z\psi_0 + \varphi(z), A_0 + g(z)\omega_0 + a(z))$ is an $\eta_{t(z)}$ -monopole. In particular,

$$\frac{1}{2}q(z\psi_0 + \varphi(z)) - *(F_{A_0} + g(z)d\omega_0 + da(z) + \eta_{t(z)}) = 0.$$

As $F_{A_0} = -\eta_0$, taking the inner product with $*\omega_0$ shows that

$$\begin{aligned} (\eta_{t(z)} - \eta_0, *\omega_0)_{L^2} &= \left(* \frac{1}{2}q(z\psi_0 + \varphi(z)) - da(z), *\omega_0 \right)_{L^2} \\ &= \left(\frac{1}{2}q(z\psi_0 + \varphi(z)), \omega_0 \right)_{L^2}, \end{aligned}$$

In the last line, we have used that $da(z)$ is orthogonal to the harmonic form $*\omega$. Invoking Proposition (I, 1.2), we find that

$$\begin{aligned} (\eta_{t(z)} - \eta_0, *\omega_0)_{L^2} &= \frac{1}{4} \left(c(\omega_0)(z\psi_0 + \varphi(z)), z\psi_0 + \varphi(z) \right)_{L^2} \\ &= \frac{1}{2}|z|^2\kappa + \frac{1}{2} \operatorname{Re} (zc(\omega_0)\psi_0, \varphi(z))_{L^2} + \frac{1}{4}(c(\omega_0)\varphi(z), \varphi(z))_{L^2}. \end{aligned}$$

Here, we employ the term κ as defined in (III, 26). Since φ vanishes of second order in 0, the right hand side of the above equation equals $\frac{1}{2}|z|^2\kappa$ up to a term which vanishes of order three at $z = 0$. From the Taylor expansion of the right hand side we thus infer that the signs of $(\eta_{t(z)} - \eta_0, *\omega_0)_{L^2}$ and κ coincide for small z . On the other hand, we have already observed in (III, 27) that $\operatorname{sgn} \kappa = \operatorname{SF}(\mathcal{D}_{A_0+r\omega_0}; |r| \ll 1)$. Putting these observations together proves (III, 28) and thus the first part of the proposition.

Given $[\psi, A, t]$ in the irreducible branch close to $[0, A_0, 0]$, we now want to compute $\varepsilon(\psi, A)$. Recall that this number is defined as the orientation transport along the family $\{T_{(s\psi, A)}\}$ associated to the affine path from $(0, A)$ to (ψ, A) . Part (i) of Lemma (II, 5.2) shows that in order to compute $\varepsilon(\psi, A)$ we can equally use the affine path from $(0, A_0)$ to (ψ, A) . This path is, however, homotopic to the path of configurations associated to the part of the irreducible branch connecting $[0, A_0, 0]$ with $[\psi, A, t]$, i.e., to the path

$$[0, z_0] \rightarrow \mathcal{C}(\sigma), \quad z \mapsto (\psi(z), A(z)),$$

with

$$(\psi(z), A(z)) := (z\psi_0, A_0 + g(z)\omega_0) + (\varphi(z), a(z)). \quad (\text{III, 29})$$

Here, z_0 is chosen in such a way that $[\psi(z_0), A(z_0), t(z_0)] = [\psi, A, t]$. Recall that $\varphi(z)$ and $a(z)$ denote the spinor and the 1-form part of $f(z, g(z))$ respectively. We now want to compute the orientation transport along

$T_z := T_{(\psi(z), A(z))}$ for $z \in [0, z_0]$. To simplify the corresponding calculation, we make the following consideration: As in the proof of Theorem (III, 3.2), we can represent the tangent space to the irreducible part of the parameterized moduli space at $[\psi, A, t]$ via

$$T_{[\psi, A, t]} \widehat{\mathcal{M}}_\eta^* \cong \ker (T_{(\psi, A)} + K_{(\psi, A, t)}) \subset L_1^2(M, E \oplus i\mathbb{R}) \oplus \mathbb{R},$$

where $K_{(\psi, A, t)}(x) := x \cdot \frac{d}{dt}|_t \text{SW}_{\eta_t}(\psi, A)$. This shows that $T_{(\psi, A)}$ is injective whenever the projection $T_{[\psi, A, t]} \widehat{\mathcal{M}}_\eta^* \rightarrow \mathbb{R}$ is an isomorphism. According to Sard's Theorem, this is true for t in a dense open subset of $[-1, 1]$ because the projection $\widehat{\mathcal{M}}_\eta^* \rightarrow [-1, 1]$ is a C^3 map between 1-dimensional manifolds.

This implies that by possibly choosing $[\psi, A, t]$ closer to $[0, A_0, 0]$ it can be guaranteed that for each $0 < z \leq z_0$, the operator $T_z := T_{(\psi(z), A(z))}$ is invertible. Therefore, the only contribution to the orientation transport along T_z is encoded in the spectral flow of T_z at $z = 0$. The latter can be understood by means of the crossing operator

$$C_T(0) := \text{Proj}_{\ker T_0} \circ \left(\frac{d}{dz} \Big|_{z=0} T_z \right) \Big|_{\ker T_0},$$

cf. Definition (C, 1.7). The operator T_z is explicitly given by

$$\begin{aligned} T_z \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} &= \begin{pmatrix} \mathcal{D}_{A_0} & 0 & 0 \\ 0 & - * d & 2d \\ 0 & 2d^* & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{1}{2}c(g(z)\omega_0 + a(z))\varphi + \frac{1}{2}c(a)\psi(z) - f\psi(z) \\ q(\psi(z), \varphi) \\ -i \text{Im} \langle \varphi, \psi(z) \rangle \end{pmatrix}. \end{aligned}$$

Note that the first term does not depend on z . According to (III, 29), we have that $\psi'(0) = \psi_0$. Furthermore, $a(z)$ and $g(z)$ vanish of second order in 0. We thus conclude

$$\left(\frac{d}{dz} \Big|_{z=0} T_z \right) \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c(a)\psi_0 - f\psi_0 \\ q(\psi_0, \varphi) \\ -i \text{Im} \langle \varphi, \psi_0 \rangle \end{pmatrix}.$$

Recalling that $\ker T_0 = \ker \mathcal{D}_{A_0} \oplus \mathcal{H}^1(M) \oplus i\mathbb{R}$, we employ real coordinates

$$(u, v, x, y) \mapsto \left((u + iv)\psi_0, x \frac{\omega_0}{\|\omega_0\|}, iy \right) \in \ker T_0.$$

The operator $\left(\frac{d}{dz} \Big|_{z=0} T_z \right) \Big|_{\ker T_0}$ is then represented by

$$(u, v, x, y) \mapsto \begin{pmatrix} \frac{1}{2}c(x \frac{\omega_0}{\|\omega_0\|})\psi_0 - iy\psi_0 \\ q(\psi_0, u\psi_0 + v i\psi_0) \\ -i \text{Im} \langle u\psi_0 + v i\psi_0, \psi_0 \rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c(x \frac{\omega_0}{\|\omega_0\|})\psi_0 - iy\psi_0 \\ uq(\psi_0) \\ -iv|\psi_0|^2 \end{pmatrix}.$$

On the other hand, the orthogonal projection $\text{Proj}_{\ker T_0}$ is given by

$$(\varphi, a, f) \mapsto \left(\text{Re}(\varphi, \psi_0)_{L^2} \psi_0 + \text{Re}(\varphi, i\psi_0)_{L^2} i\psi_0, \left(a, \frac{\omega_0}{\|\omega_0\|}\right)_{L^2} \frac{\omega_0}{\|\omega_0\|}, i\|f\|_{L^2} \right)$$

since $(\psi_0, i\psi_0, \frac{\omega_0}{\|\omega_0\|}, i)$ forms an orthonormal basis of $\ker T_0$. With respect to this basis, the operator $C_T(0)$ corresponds to

$$(u, v, x, y) \mapsto \begin{pmatrix} \text{Re}(\frac{1}{2}c(x \frac{\omega_0}{\|\omega_0\|})\psi_0, \psi_0)_{L^2} \\ \text{Re}(-iy\psi_0, i\psi_0)_{L^2} \\ (uq(\psi_0), \frac{\omega_0}{\|\omega_0\|})_{L^2} \\ -v\|\psi_0\|_{L^2} \end{pmatrix} = \begin{pmatrix} x \frac{\kappa}{\|\omega_0\|} \\ -y \\ u \frac{\kappa}{\|\omega_0\|} \\ -v \end{pmatrix}.$$

Note that we have applied Proposition (I, 1.2) to express the third row in terms of κ .

We now conclude that the crossing operator has the matrix description

$$C_T(0) \begin{pmatrix} u \\ v \\ x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\kappa}{\|\omega_0\|} & 0 \\ 0 & 0 & 0 & -1 \\ \frac{\kappa}{\|\omega_0\|} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ x \\ y \end{pmatrix}.$$

Therefore, $\det C_T(0) = \frac{\kappa^2}{\|\omega_0\|^2} > 0$ which proves that the crossing operator at the starting point of the path T_z can only have an even number of negative eigenvalues. Therefore, the spectral flow of T_z is even. According to the orientation transport formula of Theorem (C, 2.5), we may finally deduce that $\varepsilon(T_z) = 1$ which proves the claimed formula. \square

After having analysed the local structure of the parametrized moduli space near the circle of reducibles, we are now able to prove the main result of this section.

Theorem (III, 4.11) (Wall-Crossing Formula). *Let M be a closed, oriented 3-manifold with first Betti number $b_1 = 1$. Suppose that M is equipped with a spin^c structure σ and an orientation of the second cohomology. For an arbitrary metric g on M let $\eta_{\pm 1} \in \mathcal{P}_k^\pm(\sigma)$ be suitable perturbations with respect to g . Then for any path η connecting η_{-1} with η_1 and satisfying the properties of Corollary (III, 4.5),*

$$\text{sw}_{\eta_1}(\sigma) - \text{sw}_{\eta_{-1}}(\sigma) = \text{SF}(\mathcal{D}_{A_0 + r\omega_0}; r \in [0, 1]), \quad (\text{III}, 30)$$

where A_0 is an arbitrary reducible η_0 -monopole, while ω_0 is the generator of reducible η_0 monopoles. Therefore,

$$\text{sw}^+(\sigma) - \text{sw}^-(\sigma) = -\frac{1}{2} \int_M [\frac{1}{4\pi i} \omega_0] \wedge c(\sigma).$$

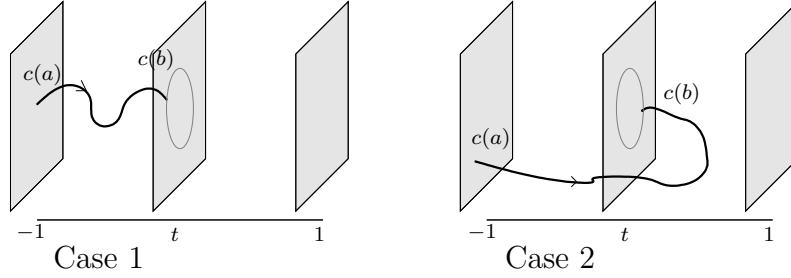


Figure III.4: The two possibilities how an arc might approach a reducible

Proof. According to part (ii) of Corollary (III, 4.5), the irreducible part of $\widehat{\mathcal{M}}_\eta(\sigma)$ is a 1-dimensional C^3 -submanifold of $\mathcal{B}^* \times [-1, 1]$. Hence, it can be written as the union of a finite number of C^3 -arcs

$$c_i : [a_i, b_i] \rightarrow \mathcal{B}^* \times [-1, 1], \quad c_i(s) = [\psi_i(s), A_i(s), t_i(s)], \quad i = 1, \dots, n.$$

As in the proof of Theorem (III, 3.2), the contribution to $\text{sw}_{\eta_1}(\sigma) - \text{sw}_{\eta_{-1}}(\sigma)$ of an arc connecting points lying in the boundary $(\mathcal{M}_{\eta_{-1}} \times \{-1\}) \cup (\mathcal{M}_{\eta_1} \times \{1\})$ is always 0. However, some endpoints may lie in the circle of reducibles (cf. Fig. III.3). We have to show that the contribution of these arcs is exactly the of the spectral flow of $\mathcal{D}_{A_0 + r\omega_0}$. Let us assume from now on that $c[a, b] \rightarrow \mathcal{B} \times [-1, 1]$ is an arc for which, say, $c(b)$ lies in the reducible part. As a consequence of Lemma (III, 4.8), this implies that $A(b)$ is degenerate. Part (iii) of Corollary (III, 4.5) guarantees, though, that $A(b)$ is only slightly degenerate. Therefore, Proposition (III, 4.10) describes the way in which way the image set of c near $c(b)$ meets the circle of reducibles. To relate $\varepsilon(\psi(a), A(a))$ with the spectral flow of $\mathcal{D}_{A(b) + r\omega_0}$ we have to account for several different situations.

Let us assume first that $c(a) \in \mathcal{M}_{\eta_{-1}}(\sigma) \times \{-1\}$. Then there are two possibilities of how c might meet the circle of reducibles (cf. Fig. III.4).

Case 1: $c(s)$ is contained in $\mathcal{B}^* \times [-1, 0)$ for s close to b : Under this assumption, we can find s_0 in any arbitrarily small neighbourhood of b such that $t'(s_0) > 0$. Exactly as in the proof of Theorem (III, 3.2), this results in

$$\varepsilon(T_{(\psi(s), A(s))}; s \in [a, s_0]) = 1.$$

In combination with homotopy invariance of the orientation transport this proves

$$\varepsilon(\psi(a), A(a)) = \varepsilon(\psi(s_0), A(s_0)).$$

On the other hand, we infer from the last assertion of Proposition (III, 4.10) that

$$\varepsilon(\psi(s_0), A(s_0)) = 1,$$

where we possibly have to adjust s_0 . Moreover, since $c(s)$ meets $[0, A(b), 0]$ coming from $\mathcal{B}^* \times [-1, 0)$, we deduce from the first part of the same result that

$$\text{SF}(\mathcal{D}_{A(b)+r\omega_0}; |r| \ll 1) = -1.$$

Combining the above observations, we conclude that

$$\varepsilon(\psi(a), A(a)) = -\text{SF}(\mathcal{D}_{A(b)+r\omega_0}; |r| \ll 1).$$

Therefore, the contribution of c to $\text{sw}_{\eta_1}(\sigma) - \text{sw}_{\eta_{-1}}(\sigma)$ is $\text{SF}(\mathcal{D}_{A(b)+r\omega_0}; |r| \ll 1)$.

Case 2: $c(s)$ is contained in $\mathcal{B}^* \times (0, 1]$ for s close to 1: Invoking Proposition (III, 4.10) in the same way as before, we find that in this case

$$\varepsilon(\psi(a), A(a)) = -\varepsilon(\psi(s_0), A(s_0)) = -1,$$

where s_0 is chosen appropriately. On the other hand,

$$\text{SF}(\mathcal{D}_{A(b)+r\omega_0}; |r| \ll 1) = 1,$$

and therefore again,

$$\varepsilon(\psi(a), A(a)) = -\text{SF}(\mathcal{D}_{A(b)+r\omega_0}; |r| \ll 1).$$

Let us now assume that c connects an element of $\mathcal{M}_{\eta_1}(\sigma) \times \{1\}$ with a reducible. Similar arguments show that independently of the direction from which c meets the circle of reducibles,

$$\varepsilon(\psi(a), A(a)) = \text{SF}(\mathcal{D}_{A(b)+r\omega_0}; |r| \ll 1)$$

so that again, the contribution to $\text{sw}_{\eta_1}(\sigma) - \text{sw}_{\eta_{-1}}(\sigma)$ is the spectral flow of $\mathcal{D}_{A(b)+r\omega_0}$ near $A(b)$.

However, we have not yet taken all possibilities into account for c might also connect two distinct reducibles $[0, A(a), 0]$ and $[0, A(b), 0]$. In this situation, there are further cases to distinguish (cf. Fig. III.5).

Case 3: $c(s) \in \mathcal{B}^* \times [-1, 0)$ for both, $s \rightarrow a$ and $s \rightarrow b$: Choose s_a and s_b appropriately close to a and b respectively. Then the last assertion of Proposition (III, 4.10) shows that

$$\varepsilon(\psi(s_a), A(s_a)) = 1 = \varepsilon(\psi(s_b), A(s_b)).$$

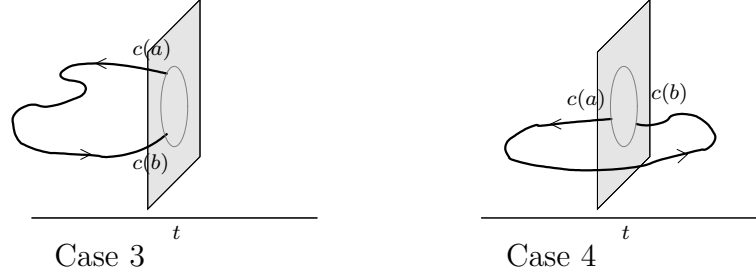


Figure III.5: Paths connecting distinct reducibles

On the other hand, transferring the arguments of Theorem (III, 3.2) to the situation at hand yields

$$\varepsilon(T_{(\psi(s), A(s))}; s \in [s_a, s_b]) = -1$$

which contradicts the above. Hence, this case can actually not occur. Clearly, the same occurs if we assume $c(s) \in \mathcal{B}^* \times (0, 1]$ for s close to a and b .

Case 4: $c(s) \in \mathcal{B}^* \times [-1, 0)$ for $s \rightarrow a$, and $c(s) \in \mathcal{B}^* \times (0, 1]$ for $s \rightarrow b$: In contrast to the above case such an arc may indeed exist. According to the first part of Proposition (III, 4.10), we then have

$$\text{SF}(\mathcal{D}_{A(a)+r\omega_0}; |r| \ll 1) = -1 = -\text{SF}(\mathcal{D}_{A(b)+r\omega_0}; |r| \ll 1).$$

Therefore, such an arc neither contributes to the left hand side nor to the right hand side of the equation we aim to prove.

These considerations show that every arc c hitting the circle of reducibles in some point $[0, A_i, 0]$ yields a summand $\text{SF}(\mathcal{D}_{A_i+r\omega_0}; |x| \ll 1)$ in the left hand side of (III, 30). Therefore,

$$\text{sw}_{\eta_1}(\sigma) - \text{sw}_{\eta_{-1}}(\sigma) = \sum_{[0, A_i, 0]} \text{SF}(\mathcal{D}_{A_i+r\omega_0}; |r| \ll 1),$$

where the sum is taken over all gauge equivalence classes $[0, A_i, 0]$ hit by an arc in the parametrized moduli space.

Letting A_0 be an arbitrary reducible η_0 -monopole, part (iii) of Corollary (III, 4.5) ensures that $\{\mathcal{D}_{A_0+r\omega_0}\}_{r \in [0, 1]}$ is a transversal family with only simple crossings. Hence, the spectral flow of $\{\mathcal{D}_{A_0+r\omega_0}\}_{r \in [0, 1]}$ is given by summing up the contributions near all slightly degenerate reducibles. These are exactly

the points where an arc in the parametrized moduli space hits the circle of reducibles. Therefore,

$$\sum_{[0, A_i, 0]} \text{SF}(\mathcal{D}_{A_i + r\omega_0}; |r| \ll 1) = \text{SF}(\mathcal{D}_{A_0 + r\omega_0}; r \in [0, 1])$$

which proves the first version of the wall-crossing formula.

We will only briefly sketch how the second version follows from the first one. Since $[\omega_0] \in H_{dR}^1(M; 4\pi i\mathbb{Z})$, it follows from (D, 15) that there exists a gauge transformation γ such that

$$[\omega_0] = [2\gamma^{-1}d\gamma].$$

This gauge transformation determines a line bundle $\hat{L}(\sigma) \rightarrow S^1 \times M$ by means of identifying the ends of the pullback of $L(\sigma)$ to $[0, 1] \times M$ via γ . The family $\{A + x\omega_0\}_{x \in [0, 1]}$ then gives rise to a connection \hat{A} on $\hat{L}(\sigma)$. It turns out that $\hat{L}(\sigma)$ is the canonical line bundle associated to the pullback⁴ of the spin^c structure σ on M to $S^1 \times M$. A result⁵ of Atiyah et. al. [3] ensures that the spectral flow of $\{\mathcal{D}_{A_0 + x\omega_0}\}_{x \in [0, 1]}$ equals the index of the spin^c Dirac operator associated to \hat{A} . This index can be computed using the Atiyah-Singer index Theorem and it turns out that (see Lim [32], Sec. 4.2)

$$\text{SF}(\mathcal{D}_{A_0 + x\omega_0}; x \in [0, 1]) = \frac{1}{8} \int_{[0, 1] \times M} \frac{i}{2\pi} F_{\hat{A}} \wedge \frac{i}{2\pi} F_{\hat{A}}.$$

Since $\hat{A} = A_0 + x\omega_0$, we compute

$$\frac{i}{2\pi} F_{\hat{A}} \wedge \frac{i}{2\pi} F_{\hat{A}} = \left(\frac{i}{2\pi} F_{A_0} + \frac{i}{2\pi} dx \wedge \omega_0 \right)^2 = 2dx \wedge \frac{i}{2\pi} \omega_0 \wedge \frac{i}{2\pi} F_{A_0}.$$

Therefore,

$$\text{SF}(\mathcal{D}_{A_0 + x\omega_0}; x \in [0, 1]) = -\frac{1}{2} \int_M \frac{1}{4\pi i} \omega_0 \wedge \frac{i}{2\pi} F_{A_0} = -\frac{1}{2} \int_M \left[\frac{1}{4\pi i} \omega_0 \right] \wedge c(\sigma). \quad \square$$

5 Manifolds with $b_1 = 0$

Let M be a closed, oriented 3-manifold with first Betti number $b_1 = 0$. Hence, $H^1(M; \mathbb{R}) = H^2(M; \mathbb{R}) = 0$. A manifold of this type is usually

⁴For a similar discussion concerning manifolds of the type $[0, 1] \times M$ see [45], Sec. 2.4.1.

⁵The theorem we are referring to seems to be rather a folklore result than a well-established fact; it is, however, well motivated in loc. cit. Moreover, Robbin & Salamon [50] give a rigorous proof in a context which slightly differs from the situation at hand.

called a *rational homology sphere* which refers to the fact that its rational cohomology or, equivalently, its real cohomology is the one of S^3 . A manifold whose singular homology equals $H_*(S^3; \mathbb{Z})$ is then called an *integer homology sphere*.

Let us first consider some specialities related to the vanishing of b_1 . First of all, the possible number of spin^c structures is very limited. Since $H^2(M; \mathbb{R}) = 0$, the image of the canonical class of a spin^c structure σ in the real cohomology always vanishes. Therefore, the canonical line bundle $L(\sigma)$ is flat. Hence, there exists up to gauge equivalence a unique flat connection, which we denote⁶ by A^b . Moreover, we have already observed in Corollary (III, 2.8) that for each $\eta \in Z_k^2(M; i\mathbb{R})$, the perturbed moduli space $\mathcal{M}_\eta(\sigma)$ contains exactly one reducible point. Up to a choice of A^b , a canonical representative of this reducible is given by $A^b - d^{-1}\eta$, where $d^{-1}\eta$ denotes the unique co-closed 1-form ω such that $d\omega = \eta$. Here, we are using that $H^1(M; \mathbb{R}) = 0$ which in association with Hodge decomposition implies that $\Omega^1(M) = \ker d \oplus \ker d^*$. Note, however, that the map d^{-1} depends on the metric.

Count of monopoles in the case $b_1 = 0$. Since the definition of $\text{sw}_\eta(\sigma)$ in Definition (III, 2.9) is not well-suited if the underlying manifold is a rational homology sphere, we have to invoke some further considerations. As before, we call a reducible η -monopole A *non-degenerate* if \mathcal{D}_A is invertible and *slightly degenerate* if \mathcal{D}_A has a one dimensional kernel.

Proposition (III, 5.1). *Let M be a rational homology 3-sphere with Riemannian metric g . Furthermore, let σ be a spin^c structure on M and let A^b be a flat connection on $L(\sigma)$. Then, for any $\eta \in Z_k^2(M; i\mathbb{R})$ such that $A^b - d^{-1}\eta$ is non-degenerate, the reducible η -monopole lies isolated in $\mathcal{M}_\eta(\sigma; g)$.*

Proof. Let $A_0 := A^b - d^{-1}\eta$. We infer from the slice theorem that a neighbourhood of $[0, A_0]$ in \mathcal{B} is homeomorphic to a neighbourhood of 0 in $(L_1^2(M, S) \oplus \ker(d^*|_{L_1^2})) / \text{U}_1$. Modulo U_1 , the moduli space near $[0, A_0]$ is then given by the zeros of

$$\begin{aligned} L_1^2(M, S) \oplus \ker(d^*|_{L_1^2}) &\rightarrow L^2(M, S) \oplus \ker d^*, \\ (\varphi, a) &\mapsto \text{Proj}_{L^2(M, S) \oplus \ker d^*} \text{SW}(\varphi, A_0 + a). \end{aligned}$$

Note that the projection onto $L^2(M, S) \oplus \ker d^*$ does not produce any new zeros. This is for the same reason as in the case of the parametrized moduli

⁶Since A^b is a 1-form, no notational confusion with the isomorphism $\flat : TM \rightarrow T^*M$ induced by the metric should be feasible.

space in Lemma (III, 4.7). The differential of the above map at the point 0 is given by

$$\begin{aligned} L_1^2(M, S) \oplus \ker(d^*|_{L_1^2}) &\rightarrow L^2(M, S) \oplus \ker d^*, \\ (\varphi, a) &\mapsto (\mathcal{D}_{A_0}\varphi, - * da). \end{aligned}$$

Its kernel is clearly equal to $\ker \mathcal{D}_{A_0} \oplus \ker d|_{\ker d^*}$. Since A_0 is non-degenerate, $\ker \mathcal{D}_{A_0} = 0$. Furthermore, $\ker d \cap \ker d^* = 0$ for rational homology spheres. Hence, the differential is invertible so that the inverse function theorem shows that 0 is an isolated point of the zero set. Therefore, $[0, A_0]$ is an isolated point of $\mathcal{M}_\eta(\sigma; g)$. \square

Definition (III, 5.2). Let M be a rational homology 3-sphere equipped with a Riemannian metric g and a spin^c structure σ .

- (i) An element $\eta \in Z_k^2(M; i\mathbb{R})$ is called a *suitable perturbation* with respect to g if the η -perturbed moduli space $\mathcal{M}_\eta(\sigma; g)$ consists only of non-degenerate points.
- (ii) If $\eta \in Z_k^2(M; i\mathbb{R})$ is a suitable perturbation with respect to g , we define

$$\text{sw}_\eta(\sigma; g) := \sum_{[\psi, A] \in \mathcal{M}_\eta^*(\sigma; g)} \varepsilon(\psi, A),$$

Note that the sum is taken over the irreducible part of the moduli space which is finite since, according to Proposition (III, 5.1), the reducible point is isolated and thus cannot be an accumulation point for irreducible monopoles. Recall from Theorem (III, 2.4) that the condition that irreducible η -monopoles are non-degenerate is a generic property. As we shall see below, this is true also for the condition that the reducible η -monopole is non-degenerate.

Finding suitable paths. Ultimately, we are going to establish is a formula describing the dependence of $\text{sw}_\eta(\sigma; g)$ on g and η . Therefore, as in the previous sections, we need to find an appropriate path connecting two suitable perturbations.

Proposition (III, 5.3). *Let M be a rational homology 3-sphere with spin^c structure σ , and let A^b be a fixed flat connection on $L(\sigma)$. If $\{g_t\}_{t \in [-1, 1]}$ is a smooth path of Riemannian metrics on M , then a generic C^m -path $a_t : [-1, 1] \rightarrow L_k^2(M, iT^*M)$ has the following properties:*

- (i) *The family $\{\mathcal{D}_{A^b + a_t}^t\}_{t \in [-1, 1]}$ is transversal with only simple crossings,*

(ii) If ψ is a non-vanishing harmonic spinor with respect to $\mathcal{D}_{A^b+a_t}^t$, then

$$(d^{-1} *_t q_t(\psi), q_t(\psi))_{L^2}^t \neq 0,$$

where the index t refers to the metric g_t .

Proof. We may consider both parts independently since the intersection of two generic sets is again generic. The proof of the first part is very similar to the corresponding proof in the case $b_1 = 1$. First of all, we fix a Riemannian metric g on M and consider again the Hilbert bundle $V \rightarrow X$ over $X := L_1^2(M, S) \setminus \{0\}$, defined by $V_\psi := \ker \operatorname{Re}(i\psi, \cdot)_{L^2}$. Define

$$\Phi : X \rightarrow V, \quad \Phi(\psi) := \mathcal{D}_{A^b}\psi.$$

This section is Fredholm of \mathbb{R} -index 1. Exactly as before, we can make this section transversal by invoking the perturbation

$$\hat{\Phi} : X \times P \rightarrow \operatorname{pr}_1^* V, \quad \hat{\Phi}(\psi, a) := \mathcal{D}_{A^b+a}\psi,$$

where $P := L_k^2(M, iT^*M)$ denotes the perturbation space.

Now let g_t be a smooth path of Riemannian metrics on M . Applying Proposition (III, 1.4), we draw the conclusion that for a generic C^m -path a_t , the set

$$\bigcup_{t \in [-1, 1]} \ker \mathcal{D}_{A_0+a_t}^t \setminus \{0\} \times \{t\}$$

is either empty or carries the structure of a 2-dimensional real C^m -submanifold of $X \times [-1, 1]$. The same arguments as in Proposition (III, 4.4) then ensure that a_t satisfies part (i) of the assertion.

Turning our attention to part (ii), we fix the metric again and consider the section

$$\Phi' : X \rightarrow V \oplus \mathbb{R}, \quad \Phi'(\psi) := \left(\mathcal{D}_{A^b}\psi, (d^{-1} * \Pi \circ q(\psi), q(\psi))_{L^2} \right),$$

where $\Pi := \operatorname{Proj}_{\ker d^*}$. The relation $(d^{-1})^* = *d^{-1}*$ together with a simple computation shows that

$$D_\psi \Phi'(\varphi) = \left(\mathcal{D}_{A^b}\varphi, (d^{-1} * \Pi \circ q(\psi), 4q(\psi, \varphi))_{L^2} \right).$$

We want to prove that $D_\psi \Phi' : L_1^2(M, S) \rightarrow \ker \operatorname{Re}(i\psi, \cdot)_{L^2} \oplus \mathbb{R}$ is a Fredholm operator of index 0. For this we consider the formal adjoint. Let $\varphi_0 \in$

$(\ker \operatorname{Re}(i\psi, \cdot)_{L^2} \cap L_1^2)$ and $r \in \mathbb{R}$. Then for any $\varphi \in L_1^2(M, S)$,

$$\begin{aligned} & \left(D_\psi \Phi'(\varphi), (\varphi_0, r) \right)_{L^2} \\ &= \operatorname{Re} \left(\mathcal{D}_{A^\flat} \varphi, \varphi_0 \right)_{L^2} + r \cdot \left(d^{-1} * \Pi \circ q(\psi), 4 \cdot q(\psi, \varphi) \right)_{L^2} \\ &= \operatorname{Re} \left(\varphi, \mathcal{D}_{A^\flat} \varphi_0 + 2r \cdot c(d^{-1} * \Pi \circ q(\psi))\psi \right)_{L^2}, \end{aligned}$$

where we have used the formula $(a, q(\psi, \varphi))_{L^2} = \frac{1}{2} \operatorname{Re} (c(a)\psi, \varphi)_{L^2}$ of Proposition (I, 1.2). Therefore, the formal adjoint of $D_\psi \Phi'$ is given by

$$\begin{aligned} & (\ker \operatorname{Re}(i\psi, \cdot)_{L^2} \cap L_1^2) \oplus \mathbb{R} \rightarrow L^2(M, S), \\ & (\varphi_0, r) \mapsto \mathcal{D}_{A^\flat} \varphi_0 + 2r \cdot c(d^{-1} * \Pi \circ q(\psi))\psi. \end{aligned}$$

The first summand is a Fredholm operator of index 0. Since the second term is compact, we deduce that the formal adjoint of $D_\psi \Phi'$ is Fredholm of index 0. Therefore, $D_\psi \Phi'$ is also Fredholm of index 0.

We now proceed in the spirit of the perturbation results in Section 1 and make Φ' transversal to the zero section. Let

$$\hat{\Phi}' : X \times L_k^2(M, iT^*M) \rightarrow V \oplus \mathbb{R}$$

be defined by

$$\hat{\Phi}'(\psi, a) := \Phi'(\psi) + \left(\frac{1}{2}c(a)\psi, 0 \right).$$

Since $\mathcal{D}_{A^\flat+a}\psi = 0$ at a zero (ψ, a) of $\hat{\Phi}'$, we deduce from Proposition (I, 2.1) that $q(\psi)$ is co-closed, i.e., $\Pi \circ q(\psi) = q(\psi)$. The differential of $\hat{\Phi}'$ is then given by

$$D_{(\psi, a)} \hat{\Phi}'(\varphi, b) = \left(\mathcal{D}_{A^\flat+a} \varphi + \frac{1}{2}c(b)\psi, (d^{-1} * q(\psi), 4q(\psi, \varphi))_{L^2} \right).$$

We claim that this map is surjective. Let (φ_0, r) be L^2 -orthogonal to the image of $D_{(\psi, a)} \hat{\Phi}'$. This implies that

$$\mathcal{D}_{A^\flat+a} \varphi_0 + 2r \cdot c(d^{-1} * q(\psi))\psi = 0 \quad \text{and} \quad \frac{1}{2} \operatorname{Re} (c(b)\psi, \varphi_0)_{L^2} = 0$$

for any $b \in L_k^2(M, iT^*M)$. As in the proof of Proposition (III, 2.3), the latter equation implies that there exists $f \in L_1^2(M, i\mathbb{R})$ with $\varphi_0 = f\psi$. Inserting this in the first equation then shows that

$$c(df + 2r \cdot d^{-1} * q(\psi))\psi = 0.$$

By virtue of the unique continuation principle, ψ is nowhere vanishing on a dense open subset of M . Hence, necessarily $df + 2r \cdot d^{-1} * q(\psi) = 0$. As the first

summand of left hand side is closed and the second summand is co-closed, we infer that both terms vanish. Therefore, f is constant and $r \cdot d^{-1} * q(\psi) = 0$. The first fact implies $\varphi_0 \equiv 0$ because $\text{Re}(i\psi, f\psi)_{L^2} = \text{Re}(i\psi, \varphi_0)_{L^2} = 0$. For the second term note that $q(\psi)$ is zero only at points where ψ vanishes. Therefore, $d^{-1} * q(\psi)$ cannot vanish everywhere which implies that $r = 0$.

To finish the proof of part (ii), we now consider an arbitrary smooth path g_t of Riemannian metrics on M . An m -times continuously differentiable path $a_t : [-1, 1] \rightarrow L_k^2(M, iT^*M)$ defines the parametrized zero set

$$\bigcup_{t \in [-1, 1]} \hat{\Phi}'_{g_t}(\cdot, a_t)^{-1}(0) \times \{t\}.$$

Proposition (III, 1.4) guarantees that for a generic choice of such path, this set is either empty or a 1-dimensional submanifold of $X \times [-1, 1]$. Part (ii) may now be proved by contradiction. Assume that there exists a tuple (ψ, t) such that $\mathcal{D}_{A^b + a_t}^t \psi = 0$ and $(d^{-1} * q_t(\psi), q_t(\psi))_{L^2} = 0$. Clearly, every multiple of ψ by a real constant also satisfies these two equations. On the other hand, they also hold for $i\psi$ and its real multiples because $q(i\psi) = q(\psi)$. Therefore, the fibre $\hat{\Phi}'_{g_t}(\cdot, a_t)^{-1}(0)$ is at least of real dimension 2, which contradicts the fact that the parametrized zero set is 1-dimensional. \square

Corollary (III, 5.4). *Let M be a rational homology 3-sphere, endowed with a spin^c structure σ and a flat connection A^b on $L(\sigma)$. Moreover, let g_{-1} and g_1 be Riemannian metrics on M . Suppose η_{-1} and η_1 are respectively chosen suitable perturbations. Then there exist C^m -paths g_t and η_t of metrics and perturbations connecting g_{-1} with g_1 and η_{-1} with η_1 respectively and having the following properties:*

- (i) *The family $\{\mathcal{D}_{A^b - d^{-1}\eta_t}^t\}_{t \in [-1, 1]}$ is transversal with only simple crossings.*
- (ii) *$(d^{-1} * q_t(\psi), q_t(\psi))_{L^2}^t \neq 0$ whenever ψ is a non-vanishing harmonic spinor with respect to $\mathcal{D}_{A^b - d^{-1}\eta_t}^t$.*
- (iii) *The irreducible part of the parametrized moduli space $\widehat{\mathcal{M}}_\eta(\sigma; g)$ is a 1-dimensional C^m -submanifold of $\mathcal{B}^* \times [-1, 1]$.*

Proof. Let g_t be a path of Riemannian metrics connecting g_{-1} and g_1 . Proposition (III, 5.3) shows that a generic C^m -path of imaginary valued 1-forms defines a family $\{\mathcal{D}_{A^b + a_t}^t\}$ satisfying the first two properties respectively. Due to Hodge decomposition we can write

$$a_t = df_t + d^* \mu_t,$$

where $f_t : [-1, 1] \rightarrow L_k^2(M, i\mathbb{R})$ and $\mu_t : [-1, 1] \rightarrow L_k^2(M, i\Lambda^2 T^*M)$ are C^m -paths. Using the bounded inverse d^{-1} of $d : \text{im } d^* \rightarrow Z^2(M; i\mathbb{R})$, we find a path η_t of closed, imaginary valued 1-forms such that

$$d^* \mu_t = -d^{-1} \eta_t.$$

Therefore, the path a_t is gauge equivalent to $-d^{-1} \eta_t$. Making the simple but important observation that properties (i) and (ii) are preserved if we apply a path of gauge transformations to a_t , we deduce that the family $\{\mathcal{D}_{A^b - d^{-1} \eta_t}^t\}$ satisfies these properties as well. Moreover, one achieves that the paths η_t obtained in this manner form a generic subset of the set paths connecting η_{-1} and η_1 .

On the other hand, an application of Theorem (III, 2.4) as in the proof of Theorem (III, 3.2) shows that a generic path η of closed imaginary valued 2-forms satisfies property (iii). Since the intersection of two generic sets is again generic, the assertion of the proposition follows. \square

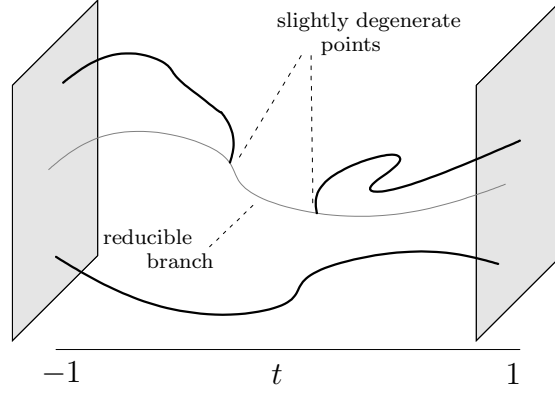
Remark. A slightly modified consideration shows that the set of all $\eta \in Z_k^2(M; i\mathbb{R})$ such that $\mathcal{D}_{A^b - d^{-1} \eta}$ is invertible forms a generic subset: From the proof of Proposition (III, 5.3), we know that a generic choice of $a \in L_{k+1}^2(M, iT^*M)$ gives rise to an invertible operator $\mathcal{D}_{A^b + a}$. Possibly applying a gauge transformation, we may assume that $a = -d^{-1} \eta$ for some closed, imaginary valued 2-form η .

Local structure of the parametrized moduli space. Fixing a C^m -path η with $m \geq 2$ as in the above corollary, we will now make a similar analysis as in Section 4.

The parametrized moduli space consists of a finite union of arcs, one of which is the reducible branch parametrized by $[0, A^b - d^{-1} \eta_t, t]$. The irreducible part forms a 1-dimensional C^m -submanifold of $\mathcal{B}^* \times [-1, 1]$, and singularities occur whenever an irreducible arc meets the reducible one (cf. Fig. III.6). We thus need to understand the local structure of $\widehat{\mathcal{M}}_\eta(\sigma; g)$ near a reducible point $[0, A^b - d^{-1} \eta_{t_0}, t_0]$. Without loss of generality, we may assume in the following that $t_0 = 0$. We then define $A_0 := A^b - d^{-1} \eta_0$.

Using the notation of Section D.4, we fix g_0 as a reference metric and employ the isometries $\hat{k}_t : L^2(M, T^*M; g_0) \rightarrow L^2(M, T^*M; g_t)$ and $\hat{\kappa}_t : L^2(M, S; g_0) \rightarrow L^2(M, S; g_t)$ to identify configurations associated to different metrics. According to the slice theorem, a neighbourhood of $[0, A_0, 0]$ in $\mathcal{B} \times [-1, 1]$ is homeomorphic to $U/U_1 \times (-\varepsilon, \varepsilon)$, where U is a U_1 -invariant open subset of $L_1^2(M, S; g_0) \times \ker(d^*|_{L_1^2})$.⁷ Modulo U_1 , the parametrized

⁷For notational convenience, we are dropping the reference to the metric g_0 in the adjoint of d .

Figure III.6: The parametrized moduli space in the case $b_1 = 0$

moduli space near $[0, A_0, 0]$ is then readily seen to be given by elements $(\varphi, a, t) \in U \times (-\varepsilon, \varepsilon)$ satisfying

$$(\mathcal{D}_{A^b - d^{-1}\eta_t + a}^t \varphi, q^t(\hat{\kappa}_t \varphi) - *_t da) = 0. \quad (\text{III}, 31)$$

As in the analogous situation in Section 4, we now define

$$s : L_1^2(M, S) \oplus \ker(d^*|_{L_1^2}) \times (-\varepsilon, \varepsilon) \longrightarrow L^2(M, S) \oplus \ker d^*$$

by letting

$$s(\varphi, a, t) := \text{Proj}_{L^2(M, S) \oplus \ker d^*} \left(\mathcal{D}_{A^b - d^{-1}\eta_t + a}^t \varphi, \hat{k}_t^{-1}(q^t(\hat{\kappa}_t \varphi) - *_t da) \right).$$

Observe that we have to employ the isometry \hat{k}_t^{-1} since $q^t(\hat{\kappa}_t \varphi) - *_t da \in \ker d^{*t}$ for any solution of (III, 31) but not necessarily $q^t(\hat{\kappa}_t \varphi) - *_t da \in \ker d^*$. Formula (D, 27) shows that \hat{k}_t induces an isomorphism $\ker d^* \rightarrow \ker d^{*t}$, and we deduce exactly in the same manner as before that the solutions of (III, 31) coincide with the zeros of s .

We shall need the differential

$$D_0 s : L_1^2(M, S) \oplus \ker(d^*|_{L_1^2}) \oplus \mathbb{R} \rightarrow L^2(M, S) \oplus \ker d^*$$

of s at the point 0 in order to invoke the implicit function theorem. A short computation yields

$$D_0 s(\varphi, a, t) = \text{Proj}_{L^2(M, S) \oplus \ker d^*} (\mathcal{D}_{A_0} \varphi, - * da) = (\mathcal{D}_{A_0} \varphi, - * da)$$

Since $\ker d^* \cap \ker d = 0$ on a rational homology sphere, we readily infer that

$$\begin{aligned} \ker D_0 s &= \ker \mathcal{D}_{A_0} \oplus \{0\} \oplus \mathbb{R} \\ \text{coker } D_0 s &= \ker \mathcal{D}_{A_0} \oplus \{0\}. \end{aligned}$$

Very similar as before we then deduce the following.

Lemma (III, 5.5). *If A_0 is non-degenerate, then the parametrized moduli space near $[0, A_0, 0]$ is locally homeomorphic to the reducible branch*

$$\bigcup_{t \in [-1, 1]} [0, A^\flat - d^{-1}\eta_t, t].$$

Proposition (III, 5.6). *Suppose that A_0 is slightly degenerate, and let $\psi_0 \in L_1^2(M, S)$ be a spinor of norm 1 spanning $\ker \mathcal{D}_{A_0}$. Then in a neighbourhood of 0 in $L_1^2(M, S) \oplus \ker(d^*|_{L_1^2}) \times (-\varepsilon, \varepsilon)$, the following holds:*

$$s(\varphi, a, t) = 0 \iff (\varphi, a, t) = \begin{cases} (0, 0, t) & \text{if } \varphi = 0 \\ (z\psi_0, 0, h(z)) + f(z, h(z)) & \text{if } \varphi \neq 0 \end{cases},$$

where

$$f : \mathbb{C} \times \mathbb{R} \rightarrow (\ker \mathcal{D}_{A_0} \oplus \mathbb{R})^\perp$$

is a U_1 -equivariant C^m -map, and $h : \mathbb{C} \rightarrow \mathbb{R}$ is a U_1 -invariant C^{m-2} -map. Both maps vanish of second order in 0.

Proof. We apply the Kuranishi technique again. Let

$$\Phi := \text{Proj}_{\text{im } D_0 s} \circ s \quad \text{and} \quad \Psi := \text{Proj}_{\text{coker } D_0 s} \circ s.$$

Transferring the arguments from the case $b_1 = 1$, we obtain a U_1 -equivariant C^m -map

$$f : \ker \mathcal{D}_{A_0} \oplus \mathbb{R} \rightarrow (\ker \mathcal{D}_{A_0} \oplus \mathbb{R})^\perp$$

such that

$$\Phi(\varphi, a, t) = 0 \iff (\varphi, a, t) = (z\psi_0, 0, t) + f(z, t),$$

where we are using coordinates $z \in \mathbb{C}$ with respect to ψ_0 . The Kuranishi obstruction map is then given by

$$\hat{\Psi} : \mathbb{C} \times \mathbb{R} \rightarrow \ker \mathcal{D}_{A_0}, \quad \hat{\Psi}(z, t) := \Psi((z\psi_0, 0, t) + f(z, t)).$$

Since $t \mapsto A^\flat - d^{-1}\eta_t$ parametrizes the reducible part of $\widehat{\mathcal{M}}_\eta(\sigma; g)$, we find that $s(0, 0, t) = 0$ and hence also $f(0, t) = \hat{\Psi}(0, t) = 0$. As before, we thus study

$$\hat{\Psi}(z, t) = z \cdot \hat{\Psi}_1(z, t), \quad \text{where } \hat{\Psi}_1(z, t) := \begin{cases} \frac{\hat{\Psi}(z, t)}{z}, & z \neq 0, \\ \left. \frac{\partial}{\partial z} \right|_{(0, t)} \hat{\Psi}(z, t), & z = 0. \end{cases}$$

As $\hat{\Psi}$ is a C^m -map, the function $\hat{\Psi}_1$ is at least C^{m-2} . Before applying the implicit function theorem again, we have to ascertain that the t derivative of this function does not vanish in $(0, 0)$. Letting $\Pi := \text{Proj}_{\ker \mathcal{D}_{A_0}}$, we compute

$$\begin{aligned} \frac{\partial}{\partial z} \Big|_{(0,t)} \hat{\Psi}(z, t) &= \Pi \left(\frac{\partial}{\partial z} \Big|_{(0,t)} s(z\psi_0, 0, t) \right) + \Pi \left(\frac{\partial}{\partial z} \Big|_{(0,t)} (s \circ f) \right) \\ &= (\mathcal{D}_{A^b-d^{-1}\eta_t}^t \psi_0, \psi_0)_{L^2} \cdot \psi_0 + \left(\frac{\partial}{\partial z} \Big|_{(0,t)} \mathcal{D}_{A_0+a(z,t)} \varphi(z, t), \psi_0 \right)_{L^2} \cdot \psi_0, \end{aligned}$$

where $\varphi(z, t)$ and $a(z, t)$ denote the spinor and the 1-form part of $f(z, t)$ respectively. With exactly the same arguments as in the proof of Proposition (III, 4.9), one deduces that the second term in the above equation's last line vanishes. Thus,

$$\begin{aligned} \text{sgn} \left(\frac{\partial}{\partial t} \Big|_{(0,0)} \hat{\Psi}_1(z, t), \psi_0 \right)_{L^2} &= \text{sgn} \left(\frac{\partial^2}{\partial t \partial z} \Big|_{(0,0)} \hat{\Psi}(z, t), \psi_0 \right)_{L^2} \\ &= \text{sgn} \left(\frac{d}{dt} \Big|_{t=0} \mathcal{D}_{A^b-d^{-1}\eta_t}^t \psi_0, \psi_0 \right)_{L^2} \\ &= \text{SF} \left(\mathcal{D}_{A^b-d^{-1}\eta_t}^t; |t| \ll 1 \right) = \pm 1, \end{aligned} \tag{III, 32}$$

where we are using that $\{\mathcal{D}_{A^b-d^{-1}\eta_t}^t\}$ has transversal spectral flow with only simple crossings. In particular, $\frac{\partial}{\partial t} \Big|_{(0,0)} \hat{\Psi}_1(z, t) \neq 0$. Hence, the implicit function theorem produces a U_1 -invariant C^{m-2} -map $h : \mathbb{C} \rightarrow \mathbb{R}$ such that in a neighbourhood of $(0, 0)$,

$$\hat{\Psi}_1(z, t) = 0 \iff t = h(z).$$

Very similar to the situation in the preceding chapter, we then infer that in a neighbourhood of 0, the claimed condition holds. The arguments concerning the property that f and h vanish of second order in 0 are also the same as before. \square

Remark. As a result, we find that a neighbourhood of $[0, A_0, 0]$ in $\widehat{\mathcal{M}}_\eta(\sigma)$ is homeomorphic to a neighbourhood of 0 in

$$\{(z, x) \in \mathbb{R}_+^0 \times \mathbb{R} \mid z = 0 \text{ or } x = 0\},$$

where $\mathbb{R}_+ \times \{0\}$ corresponds to the irreducible part. The branch $\{0\} \times \mathbb{R}$ corresponds to the reducible arc $[0, A^b - d^{-1}\eta_t, t]$.

According to Proposition (III, 5.6), the t -component of the irreducible branch near $[0, A_0, 0]$ is given by the value of h , which we shall henceforth regard as a function $\mathbb{R}_+^0 \rightarrow \mathbb{R}$. Therefore, h encodes information about on which side of $\mathcal{B} \times \{0\}$ the irreducible branch near $[0, A_0, 0]$ is located. Since h is a C^{m-2} -function vanishing of second order in 0, it is promising to assume

that $m \geq 4$ and study the second derivative of h : If $h''(0) < 0$, then h has a maximum in 0 so that the irreducible branch near $[0, A_0, 0]$ is contained in $\mathcal{B} \times [-1, 0)$. If $h''(0) > 0$, then this branch lies in $\mathcal{B} \times (0, 1]$.

The next result shows how to relate this with the spectral flow $\text{SF}(\mathcal{D}_{A^b-d^{-1}\eta_t}; |x| \ll 1)$ and the number $(d^{-1} * q(\psi_0), q(\psi_0))_{L^2}$. Recall that both numbers do not vanish according to the choice of η .

Proposition (III, 5.7). *Assume that $m \geq 4$. Then the second derivative of h is given by the formula*

$$h''(0) \cdot \left(\frac{d}{dt} \Big|_{t=0} \mathcal{D}_{A^b-d^{-1}\eta_t}^t \psi_0, \psi_0 \right)_{L^2} = -2(d^{-1} * q(\psi_0), q(\psi_0))_{L^2}$$

and thus never equals 0. Moreover, if $m \geq 6$ and if $[\psi, A, t]$ is an element of the irreducible branch close to $[0, A_0, 0]$, then

$$\varepsilon(\psi, A) = \text{sgn } h''(0) \cdot \text{SF}(\mathcal{D}_{A^b-d^{-1}\eta_t}; |t| \ll 1).$$

Proof. Let us write $f(z, h(z)) = (\varphi(z), a(z), 0)$. The irreducible branch is then locally parametrized by

$$(0, z_0) \rightarrow \mathcal{B} \times [-1, 1], \quad z \mapsto [\psi(z), A(z), h(z)],$$

where $z_0 \in \mathbb{R}_+$ is appropriately small, and where

$$\psi(z) := z\psi_0 + \varphi(z) \quad \text{and} \quad A(z) := A^b - d^{-1}\eta_{h(z)} + a(z).$$

According to the above proposition, the path $(\psi(z), A(z), h(z))$ can be extended to $z = 0$ in a twice continuously differentiable way since we have chosen $m \geq 4$.

We differentiate the Dirac equation for $(\psi(z), A(z))$ two times and obtain

$$0 = \frac{d^2}{dz^2} \left(\mathcal{D}_{A(z)}^{h(z)} \right) \psi(z) + 2 \frac{d}{dz} \left(\mathcal{D}_{A(z)}^{h(z)} \right) \psi'(z) + \mathcal{D}_{A(z)}^{h(z)} \psi''(z).$$

If we now take the inner product with ψ_0 , then a careful analysis shows that we can differentiate the resulting equation at $z = 0$ once again. For this we observe by making use of self-adjointness of $\mathcal{D}_{A(z)}^{h(z)}$ and the chain rule

$$\begin{aligned} \frac{1}{z} \left(\mathcal{D}_{A(z)}^{h(z)} \psi''(z), \psi_0 \right)_{L^2} &= \left(\psi''(z), \frac{1}{z} \mathcal{D}_{A(z)}^{h(z)} \psi_0 \right)_{L^2} \\ &\xrightarrow{z \rightarrow 0} \left(\psi''(0), h'(0) \frac{d}{dt} \Big|_{t=0} \mathcal{D}_{A^b-d^{-1}\eta_t}^t \psi_0 + c(a'(0)) \psi_0 \right)_{L^2} = 0 \end{aligned}$$

because a and h vanish of second order. Moreover,

$$\frac{1}{z} \left(\frac{d^2}{dz^2} \left(\mathcal{D}_{A(z)}^{h(z)} \right) \psi(z), \psi_0 \right)_{L^2} \xrightarrow{z \rightarrow 0} \left(\frac{d^2}{dz^2} \Big|_{z=0} \left(\mathcal{D}_{A(z)}^{h(z)} \right) \psi'(0), \psi_0 \right)_{L^2}.$$

Thus, the chain rule implies that

$$\begin{aligned} 0 &= \left(\frac{d^2}{dz^2} \Big|_{z=0} (\mathcal{D}_{A(z)}^{h(z)}) \psi'(0), \psi_0 \right)_{L^2} \\ &= \left(\frac{d}{dz} \Big|_{z=0} \left(h'(z) \frac{d}{dt} \Big|_{t=h(z)} \mathcal{D}_{A^b-\eta_t}^t + \frac{1}{2} c(a'(z)) \right) \psi'(0), \psi_0 \right)_{L^2} \\ &= \left(h''(0) \frac{d}{dt} \Big|_{t=0} \mathcal{D}_{A^b-\eta_t}^t \psi_0 + \frac{1}{2} c(a''(0)) \psi_0, \psi_0 \right)_{L^2}, \end{aligned}$$

where we invoke that $\psi'(0) = \psi_0$ and that h and a vanish of second order in 0. Therefore,

$$h''(0) \cdot \left(\frac{d}{dt} \Big|_{t=0} \mathcal{D}_{A^b-\eta_t}^t \psi_0, \psi_0 \right)_{L^2} = -\frac{1}{2} (c(a''(0)) \psi_0, \psi_0)_{L^2}.$$

The term on the right hand side can be computed by making use of the second part of the Seiberg-Witten equations. Since

$$q_{h(z)}(\hat{\kappa}_{h(z)}\psi(z)) = *_h(z)da(z),$$

differentiating twice and using that $\psi(0) = 0$ and $a(0) = a'(0) = 0$ yields:

$$2q(\psi_0) = *da''(0), \quad \text{i.e.,} \quad a''(0) = 2d^{-1} * q(\psi_0).$$

Combining this with the above result and invoking Proposition (I, 1.2), we infer that

$$h''(0) \cdot \left(\frac{d}{dt} \Big|_{t=0} \mathcal{D}_{A^b-\eta_t}^t \psi_0, \psi_0 \right)_{L^2} = -2(d^{-1} * q(\psi_0), q(\psi_0))_{L^2}.$$

This proves the first assertion.

We shall now determine the number $\varepsilon(\psi, A)$ for an irreducible η_t -monopole such that $[\psi, A, t]$ lies on the irreducible branch close to $[0, A_0, 0]$. As in the corresponding situation before, this amounts to compute the spectral flow of the family

$$T_z := T_{(\psi(z), A(z))}^{h(z)} \quad z \in [0, z_0],$$

where we suppose that $(\psi(z_0), A(z_0), h(z_0)) = (\psi, A, t)$. In addition, we may assume that T_z is invertible for any $z \neq 0$. We write

$$T_z = D_z + K_z,$$

where D_z and K_z are given by

$$D_z \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{A^b-d^{-1}\eta_{h(z)}}^{h(z)} \varphi \\ -*_h(z) da + 2df \\ 2d*_h(z)a \end{pmatrix}, \quad \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} \in L_1^2(M, E \oplus i\mathbb{R})$$

and

$$K_z \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c(a(z))\varphi + \frac{1}{2}c(a)\psi(z) - f\psi(z) \\ q_{h(z)}(\psi(z), \varphi) \\ -i \operatorname{Im} \langle \varphi, \psi(z) \rangle_{h(z)} \end{pmatrix}, \quad \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} \in L_1^2(M, E \oplus i\mathbb{R})$$

respectively. As $h(z)$ vanishes of second order in 0, we infer from the chain rule that

$$\left. \frac{d}{dz} \right|_{z=0} D_z = 0.$$

Since $\psi'(0) = \psi_0$ and $a'(0) = 0$, this results in:

$$\left(\left. \frac{d}{dz} \right|_{z=0} T_z \right) \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c(a)\psi_0 - f\psi_0 \\ q(\psi_0, \varphi) \\ -i \operatorname{Im} \langle \varphi, \psi_0 \rangle \end{pmatrix}, \quad \begin{pmatrix} \varphi \\ a \\ f \end{pmatrix} \in L_1^2(M, E \oplus i\mathbb{R}). \quad (\text{III}, 33)$$

Next observe that

$$\ker T_0 = \ker \mathcal{D}_{A_0} \oplus \{0\} \oplus i\mathbb{R} = \operatorname{Span}_{\mathbb{R}}\{\psi_0, i\psi_0, i\}.$$

If we employ real coordinates x_1, x_2 and y with respect to the above orthonormal basis, then a very similar computation as in the proof of Proposition (III, 4.10) shows that the crossing operator $C_T(0) = \operatorname{Proj}_{\ker T_0} \circ \left(\left. \frac{d}{dz} \right|_{z=0} T_z \right) |_{\ker T_0}$ has the matrix description

$$C_T(0) \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix}. \quad (\text{III}, 34)$$

We conclude that $\operatorname{spec}(C_T(0)) = \{-1, 0, 1\}$.

In association with Kato's Selection Theorem (C, 1.6), this corresponds to the following situation: We can choose three C^1 -paths $\theta_i : [0, z_0] \rightarrow \mathbb{R}$, $i \in \{-1, 0, 1\}$ parametrizing the eigenvalues of T_z which equal zero for $z = 0$. The structure of the crossing operator then shows that $\theta'_i(0) = i$. In particular, the eigenvalue θ_0 vanishes of second order in 0 so that we cannot compute the spectral flow by means of the crossing operator. We thus need a more refined analysis of θ_0 's behaviour near 0.

Remark. In the literature, the path of operators T_z is usually assumed to be analytically. One then finds an analytic parametrization of eigenvalues and corresponding eigenvectors (cf. [11], [43]). However, the author of this thesis does not see how to achieve analyticity of T . Fortunately, the considerations in Appendix C provide a way out of this trouble if we choose $m \geq 6$. Nevertheless, the involved computations remain essentially the same.

According to Example (C, 1.5), we may assume that θ_0 is twice continuously differentiable, if $m \geq 6$. Moreover, we find a C^2 -family $v_z \in L_1^2(M, E \oplus i\mathbb{R})$ locally corresponding to the eigenvectors associated to θ_0 . Formula (C, 1) then shows that

$$\begin{aligned}\theta_0''(0) &= \frac{d}{dz}\Big|_{z=0} \operatorname{Re} \left(\left(\frac{d}{dz}\Big|_z T_z \right) v_z, v_z \right)_{L^2} \\ &= \operatorname{Re} (T_0'' v_0 + T_0' v_0', v_0)_{L^2} + \operatorname{Re} (T_0' v_0, v_0')_{L^2} \\ &= \operatorname{Re} (T_0'' v_0 + 2T_0' v_0', v_0)_{L^2},\end{aligned}\tag{III, 35}$$

where we are using obvious abbreviations and invoke self-adjointness of T_0' .

First of all, note that v_0 lies in the kernel of T_0 . Letting $P := \operatorname{Proj}_{\ker T_0}$, we thus have $v_0 = P v_0$. Moreover, differentiating the equation $T_z v_z = \theta_0(z) v_z$ implies that

$$T_0' v_0 + T_0 v_0' = \theta_0'(0) v_0 + \theta_0(0) v_0' = 0$$

for $\theta_0 = \theta_0' = 0$. From this and the fact that $PT_0 = 0$ we deduce

$$0 = PT_0' v_0 + PT_0 v_0' = PT_0' P v_0 = C_T(0) v_0.$$

Therefore, $v_0 \in \ker C_T(0)$. As a consequence of (III, 34), the kernel of the crossing operator is given by $\operatorname{Span}_{\mathbb{R}} \psi_0$. Henceforth, we may thus assume that $v_0 = \psi_0$. To determine v_0' we deduce from the above computations and the explicit formula of T_0' in (III, 33) that

$$-T_0 v_0' = T_0' v_0 = T_0' \psi_0 = \begin{pmatrix} 0 \\ q(\psi_0) \\ 0 \end{pmatrix}.$$

Recalling that T_0 is explicitly given by

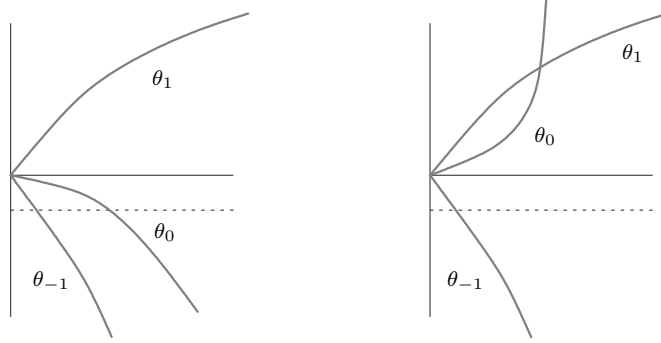
$$T_0 = \begin{pmatrix} \mathcal{D}_{A_0} & 0 & 0 \\ 0 & - * d & 2d \\ 0 & 2d^* & 0 \end{pmatrix},$$

we infer that

$$v_0' \in d^{-1} * q(\psi_0) + \ker T_0, \quad \text{i.e., } v_0' = \begin{pmatrix} z\psi_0 \\ d^{-1} * q(\psi_0) \\ iy \end{pmatrix}$$

for suitable $z \in \mathbb{C}$ and $y \in \mathbb{R}$. According to (III, 33), we thus find the following:

$$\begin{aligned}\operatorname{Re} (T_0' v_0', v_0)_{L^2} &= \operatorname{Re} \left(\frac{1}{2} c(d^{-1} * q(\psi_0)) \psi_0 - iy \psi_0, \psi_0 \right)_{L^2} \\ &= (d^{-1} * q(\psi_0), q(\psi_0))_{L^2}.\end{aligned}$$

Figure III.7: Spectral flow of T_z

Considering the second term in (III, 35), we immediately deduce from the definition of T_z that

$$\operatorname{Re} (T_0'' v_0, v_0)_{L^2} = \operatorname{Re} \left(\frac{d^2}{dz^2} \Big|_{z=0} \left(\mathcal{D}_{A(z)}^{h(z)} \right) \psi_0, \psi_0 \right)_{L^2}.$$

As a result of the computations performed in the first part of this proof, this term equals zero. Hence, we finally draw the following conclusion:

$$\begin{aligned} \theta_0''(0) &= 2 \operatorname{Re} (T_0' v_0', v_0)_{L^2} = 2 (d^{-1} * q(\psi_0), q(\psi_0))_{L^2} \\ &= -h''(0) \cdot \left(\frac{d}{dt} \Big|_{t=0} \mathcal{D}_{A^b - \eta_t}^t \psi_0, \psi_0 \right)_{L^2}, \end{aligned}$$

where we have employed the first part of this result.

The spectral flow of T_z in 0 is now obtained in the following way (cf. Fig. III.7): If $\theta_0''(0) > 0$, then the function θ has a minimum in 0. Therefore, the only eigenvalue leaving 0 in the negative direction is θ_{-1} . Due to our convention of counting eigenvalues at the endpoints, the spectral flow of T_z then equals 1. In accordance with the orientation transport formula in Theorem (C, 2.5), we conclude that $\varepsilon(\psi, A) = -1$. If, on the other hand, $\theta_0''(0) < 0$, then the eigenvalue $\theta_0(z)$ is also negative for small $z > 0$. It thus contributes to the spectral flow of T_z , and we find that $\varepsilon(\psi, A) = 1$. Therefore,

$$\varepsilon(\psi, A) = -\operatorname{sgn} \theta_0''(0) = \operatorname{sgn} h''(0) \cdot \operatorname{sgn} \left(\frac{d}{dt} \Big|_{t=0} \mathcal{D}_{A^b - \eta_t}^t \psi_0, \psi_0 \right)_{L^2}.$$

This proves the assertion since the last term in the above equation equals the spectral flow of $\{\mathcal{D}_{A^b - d^{-1}\eta_t}\}$ in $t = 0$. \square

Theorem (III, 5.8). *Let M be a rational homology 3-sphere endowed with a spin^c structure σ and a flat connection A^b on $L(\sigma)$. Suppose that g_{-1} and g_1*

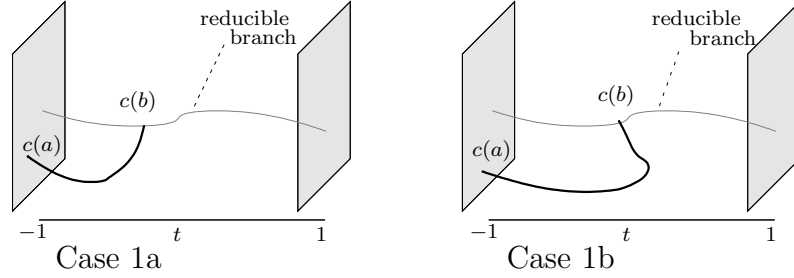


Figure III.8: An irreducible branch meeting the reducible one

be Riemannian metrics on M together with respective suitable perturbations η_{-1} and η_1 . Then

$$\text{sw}_{\eta_{-1}}(\sigma; g_{-1}) - \text{sw}_{\eta_1}(\sigma; g_1) = -\text{SF}(\mathcal{D}_{A^b-d^{-1}\eta_t}^t; t \in [-1, 1]), \quad (\text{III}, 36)$$

where g_t and η_t are arbitrary C^1 -path of metrics and perturbations connecting g_{-1} with g_1 and η_{-1} with η_1 respectively.

Proof. We will proceed exactly as in the proof of the wall-crossing formula in the preceding section. Therefore, we immediately restrict our attention to the arcs of the parametrized moduli space which meet the reducible branch. Let

$$c(s) = [\psi(s), A(s), t(s)] : [a, b] \rightarrow \mathcal{B} \times [-1, 1]$$

be an arc meeting a slightly degenerate reducible in $c(b) = [0, A(b), t(b)]$. To begin with, we make the assumption that $c(a)$ does not lie on the reducible branch (cf. Fig. III.8).

Case 1: $c(a) \in \mathcal{M}_{\eta_{-1}}(\sigma; g_{-1})$: Let us primarily assume that $t(s) < t(b)$ for s close to b . In the notation of Proposition (III, 5.6), we have $t = h(z)$, and this yields that $h''(0) < 0$. The familiar considerations of Theorem (III, 3.2) imply that for an appropriate choice of s_0 close to b , we have

$$\varepsilon(\psi(a), A(a)) = \varepsilon(\psi(s_0), A(s_0))$$

On the other hand, the fact that $h''(0) < 0$ together with Proposition (III, 5.7) shows that

$$\varepsilon(\psi(s_0), A(s_0)) = -\text{SF}(\mathcal{D}_{A^b-d^{-1}\eta_t}; |t-t(b)| \ll 1),$$

where we might possibly have to adjust s_0 a little. Hence,

$$\varepsilon(\psi(a), A(a)) = -\text{SF}(\mathcal{D}_{A^b-d^{-1}\eta_t}; |t-t(b)| \ll 1).$$

If, on the other hand, $t(s) > t(b)$ for s close to b , then

$$\varepsilon(\psi(a), A(a)) = -\varepsilon(\psi(s_0), A(s_0))$$

for appropriately chosen s_0 . Another application of Proposition (III, 5.7) implies that in this situation

$$\varepsilon(\psi(s_0), A(s_0)) = \text{SF}(\mathcal{D}_{A^b-d^{-1}\eta_t}; |t-t(b)| \ll 1).$$

We thus obtain the same formula for $\varepsilon(\psi(a), A(a))$ again.

Case 2: $c(a) \in \mathcal{M}_{\eta_1}(\sigma; g_1)$: With exactly the same arguments, one straightforwardly deduces that

$$\varepsilon(\psi(a), A(a)) = \text{SF}(\mathcal{D}_{A^b-d^{-1}\eta_t}; |t-t(b)| \ll 1)$$

irrespective of how c meets the reducible branch.

We are therefore left to consider paths connecting two points on the reducible branch.

Case 3: $t(s) < t(a)$ as $s \rightarrow a$ and $t(s) < t(b)$ as $s \rightarrow b$: Under this assumption, we find that for s_a close to a and s_b close to b ,

$$\varepsilon(\psi(s_a), A(s_a)) = -\varepsilon(\psi(s_b), A(s_b)).$$

On the other hand, the analysis of Proposition (III, 5.7) shows that for $i = a, b$,

$$\varepsilon(\psi(s_i), A(s_i)) = \text{SF}(\mathcal{D}_{A^b-d^{-1}\eta_t}; |t-t(i)| \ll 1).$$

Therefore, when studying the spectral flow of the entire path $\mathcal{D}_{A^b-d^{-1}\eta_t}$, the contributions at t_a and at t_b cancel each other out.

Case 4: $t(s) < t(a)$ if $s \rightarrow a$ and $t(s) > t(b)$ if $s \rightarrow b$: In this case,

$$\varepsilon(\psi(s_a), A(s_a)) = \varepsilon(\psi(s_b), A(s_b)).$$

However, we also have

$$\varepsilon(\psi(s_a), A(s_a)) = \text{SF}(\mathcal{D}_{A^b-d^{-1}\eta_t}; |t-t(a)| \ll 1)$$

and

$$\varepsilon(\psi(s_b), A(s_b)) = -\text{SF}(\mathcal{D}_{A^b-d^{-1}\eta_t}; |t-t(b)| \ll 1).$$

Therefore, the contributions at t_a and at t_b to the spectral flow cancel each other in this case as well.

The remaining cases are treated accordingly.

Proceeding with those paths which do not meet the reducible branch exactly as in the proof of the wall-crossing formula, the above considerations establish the claimed formula. \square

Producing an invariant for rational homology spheres. The above result shows that $\text{sw}_\eta(\sigma; g)$ depends profoundly on the metric g and the perturbation parameter η . Thus we do not obtain an invariant for rational homology spheres. Due to the resemblance with the gauge theoretical construction of the Casson invariant as performed by Taubes [52], it was soon conjectured by Kronheimer that adding an appropriate counter term to $\text{sw}_\eta(\sigma; g)$, one should obtain an invariant which on a homology sphere again equals the Casson invariant. This is indeed true as Lim [31] and Chen [11] independently proved. In the remaining part of this thesis, we will only give a formula for the counter term, briefly motivating why the resulting number is an invariant of the underlying manifold.

Let P be a first-order self-adjoint elliptic operator on a closed, oriented manifold M of odd dimension. Atiyah et al. [3] proved that the function

$$\eta_P(s) := \sum_{z \in \text{spec}(P) \setminus \{0\}} \text{sgn}(z) |z|^{-s}$$

converges for $\text{Re } s \gg 0$ and extends to a meromorphic function on the whole s -plane, with a finite value at $s = 0$. The η -invariant of P is then defined as

$$\eta_P := \eta_P(0).$$

Let σ be a spin^c structure on a rational homology 3-sphere M , and let g be a Riemannian metric with corresponding suitable perturbation $\nu \in Z_k^2(M; i\mathbb{R})$. Here, we are using the notation ν instead of η to avoid a confusion with the η -invariant. We then define $\eta_{\text{dir}}(g, \nu)$ as the η -invariant of the spin^c Dirac operator $\mathcal{D}_{A^b - d^{-1}\nu}^g$, where A^b is a fixed flat connection on $L(\sigma)$.

If g_t and ν_t are paths as in Theorem (III, 5.8), then it follows from the work of Atiyah, Patodi and Singer that

$$\begin{aligned} & \frac{1}{2} (\eta_{\text{dir}}(g_1, \nu_1) - \eta_{\text{dir}}(g_{-1}, \nu_{-1})) \\ &= \text{SF}(\mathcal{D}_{A^b - d^{-1}\nu_t}; t \in [-1, 1]) + \frac{1}{8} \int_{[-1, 1] \times M} \left(-\frac{1}{3} p_1(\hat{\nabla}) + c_1(\hat{A})^2 \right), \end{aligned}$$

where p_1 is the first Pontryagin class, and $\hat{A} = A^b - d^{-1}\nu$ and $\hat{\nabla}$ respectively denote the pullbacks of the connection on $L(\sigma)$ and the Levi-Civita covariant derivative to the corresponding bundles over $[-1, 1] \times M$. A survey of the (nontrivial) results leading to this formula can be found in Nicolaescu's

book [45], Sec. 4.1.3. Note that we have already invoked that the operator $\mathcal{D}_{A^b-d^{-1}\nu_t}$ is invertible for $t = -1, 1$ so that the dimensions of the kernels do not occur on the above formula's left hand side.

On the other hand, we have the odd signature operator on (M, g) , i.e.,

$$\mathcal{D}_{\text{sign}}^g := \begin{pmatrix} *d & -d \\ -d^* & 0 \end{pmatrix} : \Omega^1 \oplus \Omega^0 \rightarrow \Omega^1 \oplus \Omega^0.$$

This is also a first-order self-adjoint elliptic differential operator, and we denote the associated η -invariant by $\eta_{\text{sign}}(g)$. If g_t is a path of metrics, then the corresponding formula is (see Atiyah et al. [3]):

$$\eta_{\text{sign}}(g_1) - \eta_{\text{sign}}(g_{-1}) = \frac{1}{3} \int_{[-1,1] \times M} p_1(\hat{\nabla}).$$

Note that there is no spectral flow term since the kernels of $\mathcal{D}_{\text{sign}}^{g_t}$ have constant dimensions.

Considering paths g_t and ν_t as in Theorem (III, 5.8), we deduce that the term $4\eta_{\text{dir}}(g, \nu) + \eta_{\text{sign}}(g)$ behaves in the following way:

$$\begin{aligned} & 4\eta_{\text{dir}}(g_1, \nu_1) + \eta_{\text{sign}}(g_1) - 4\eta_{\text{dir}}(g_{-1}, \nu_{-1}) - \eta_{\text{sign}}(g_{-1}) \\ &= 8 \text{SF}(\mathcal{D}_{A^b-d^{-1}\nu_t}; t \in [-1, 1]) + \int_{[-1,1] \times M} c_1(\hat{A})^2 \end{aligned} \quad (\text{III, 37})$$

The second summand in the last line can be split up into a difference of two terms which only depend on ν_{-1} and ν_1 respectively. For notational convenience, we let $a_t := -d^{-1}\nu_t$. Then

$$F_{\hat{A}} = F_{A^b} + da_t + dt \wedge \frac{\partial}{\partial t} a_t = da_t + dt \wedge \frac{\partial}{\partial t} a_t.$$

Hence, we compute

$$\begin{aligned} F_{\hat{A}} \wedge F_{\hat{A}} &= dt \wedge da_t \wedge \frac{\partial}{\partial t} a_t + dt \wedge \frac{\partial}{\partial t} a_t \wedge da_t = dt \wedge (da_t \wedge \frac{\partial}{\partial t} a_t + \frac{\partial}{\partial t} a_t \wedge da_t) \\ &= dt \wedge (d(a_t \wedge \frac{\partial}{\partial t} a_t) + a_t \wedge \frac{\partial}{\partial t} da_t + \frac{\partial}{\partial t} a_t \wedge da_t) \\ &= dt \wedge (d(a_t \wedge \frac{\partial}{\partial t} a_t) + \frac{\partial}{\partial t} (a_t \wedge da_t)). \end{aligned}$$

Using Stoke's Theorem and performing the integration over $[-1, 1]$, one finds

$$\begin{aligned} \int_{[-1,1] \times M} c_1(\hat{A})^2 &= -\frac{1}{4\pi^2} \int_{[-1,1] \times M} dt \wedge \frac{\partial}{\partial t} (a_t \wedge da_t) \\ &= -\frac{1}{4\pi^2} \int_M (a_1 \wedge da_1 - a_{-1} \wedge da_{-1}) \\ &= -\frac{1}{4\pi^2} \int_M (d^{-1}\nu_1 \wedge \nu_1 - d^{-1}\nu_{-1} \wedge \nu_{-1}). \end{aligned}$$

A combination of this computation with formula (III, 36) of Theorem (III, 5.8) and formula (III, 37) easily establishes the following result:

Theorem (III, 5.9). *Let M be a rational homology 3-sphere with spin^c structure σ . If ν is a suitable perturbation with respect to a Riemannian metric g , then the “modified Seiberg-Witten invariant”*

$$\lambda_\nu(\sigma; g) := \text{sw}_\nu(\sigma; g) - \frac{1}{2}\eta_{\text{dir}}(g, \nu) - \frac{1}{8}\eta_{\text{sign}}(g) - \frac{1}{32\pi^2} \int_M d^{-1}\nu \wedge \nu$$

is independent of ν and g . Therefore, it gives rise to a smooth invariant of M .

Remark.

- (i) Note that $\lambda_\nu(\sigma; g)$ is generally not integer valued. In fact, it is \mathbb{Z} -valued provided that $H_1(M; \mathbb{Z}) = 0$. Otherwise, $8h \cdot \lambda_\nu(\sigma; g) \in \mathbb{Z}$ where $h := |H_1(M; \mathbb{Z})|$ denotes the order of the first homology group (cf. Lim [32], Prop. 17).
- (ii) If M happens to be an integer homology sphere, then the cohomology group $H^2(M; \mathbb{Z})$ is trivial. Hence, there exists only one spin^c structure on M . In [31], Y. Lim establishes that the unique number obtained in this way equals the *Casson invariant* of an integer homology sphere.
- (iii) For rational homology spheres, Marcolli & Wang [37] investigate an averaged version of the modified Seiberg-Witten invariants—obtained by summing over all spin^c structures. They prove that it equals the so-called *Casson-Walker invariant*.
- (iv) On the other hand, Nicolaescu [46] shows that a combination of the Casson-Walker invariant and certain refined torsion invariants (due to V.G. Turaev) determine all modified Seiberg-Witten invariants of a rational homology sphere.

Appendix A

Elliptic Equations on Compact Manifolds

This appendix summarizes the basic notions and results we need from the theory of elliptic differential equations on compact manifolds. In Section 1, we recall the definition of Sobolev spaces on manifolds. As the Seiberg-Witten equations are nonlinear, it is not sufficient to work only in the well-known setting of the Hilbert spaces L_k^2 ; we also have to consider the Banach spaces L_k^p for arbitrary $1 \leq p < \infty$. We state the versions of Sobolev embedding and Rellich's compactness result in this more general context, and provide a Sobolev multiplication theorem as a corollary. The corresponding proofs can be found in standard references like the books of Gilbarg & Trudinger [18], Adams [1], Taylor [54], and Aubin [4]. The latter book includes the formulation on manifolds which we need.

Section 2 is dedicated to differential operators acting on sections of vector bundles. Basically, we will only list the fundamental properties of elliptic partial differential operators and refer for most proofs to the wide range of literature (e.g. Gilbarg & Trudinger [18]). Brief expositions, yet including proofs, can also be found in many books on differential geometry (e.g. Warner [57] or Nicolaescu [44]).

1 Sobolev spaces

Let (M, g) be a closed¹ and oriented Riemannian manifold. The volume form dv_g induces a Lebesgue measure on M . For each $1 \leq p < \infty$, we can thus define the space $L^p(M, \mathbb{K})$ of (equivalence classes of) \mathbb{K} -valued measurable

¹As in the main part of the thesis we use the convention that a closed manifold is compact, connected, and has no boundary.

functions f on M for which

$$\|f\|_p := \left(\int_M |f|^p dv_g \right)^{1/p} < \infty.$$

Suppose $\pi : E \rightarrow M$ is a Hermitian or Euclidean vector bundle endowed with a connection ∇ which is compatible with the metric. We let $L^p(M, E)$ denote the space of L^p -sections of E , i.e., the space of (equivalence classes of) measurable maps $u : M \rightarrow E$ which satisfy $\pi \circ u = \text{id}_M$ almost everywhere and $|u| \in L^p(M, \mathbb{R})$. For each $k \in \mathbb{N}$ the Sobolev space $L_k^p(M, E)$ consists of all sections $u \in L^p(M, E)$ for which there exists $v \in L^p(M, T^*M^{\otimes m} \otimes E)$ such that for all $w \in C^\infty(M, T^*M^{\otimes m} \otimes E)$ and any $m \leq k$,

$$\int_M \langle v, w \rangle dv_g = \int_M \langle u, (\nabla^m)^t w \rangle dv_g$$

Here,

$$(\nabla^m)^t : C^\infty(M, T^*M^{\otimes m} \otimes E) \rightarrow C^\infty(M, E)$$

denotes the formal adjoint of ∇^m (cf. also Section 2 below). Then v is called the *weak m -th covariant derivative* of u and is denoted by $\nabla^m u$. Note that it is always defined as a distribution. Therefore, the above can be reformulated by saying

$$u \in L_k^p(M, E) \iff \nabla^m u \in L^p(T^*M^{\otimes m} \otimes E).$$

Each $L_k^p(M, E)$ is a Banach space with respect to the norm

$$\|u\|_{L_k^p} := \sum_{m \leq k} \|\nabla^m u\|_{L^p} = \sum_{m \leq k} \left(\int_M |\nabla^m u|^p dv_g \right)^{1/p}.$$

Moreover, $L_k^p(M, E)$ lies dense in $L^p(M, E)$ and contains the smooth functions as a dense subspace with respect to the norm $\|\cdot\|_{L_k^p}$ (cf. [4], Thm.2.4). Furthermore, it turns out that compactness of M guarantees that the definition of $L_k^p(M, E)$ is independent of all choices made. Any choice of different metrics and connections yields equivalent norms (cf. Aubin [4], Thm.2.20).

Remark. If $p \neq 2$, the spaces L_k^p cannot conveniently be defined for by making use of the Fourier transformation as in the case of L_k^2 . Introducing L_s^p for every $s \in \mathbb{R}$ requires interpolation theory (cf. Taylor [54], Sec. 13.6).

Embedding theorems. There are two very important results relating Sobolev spaces both with each other and with spaces of C^r functions. These

theorems are generalizations of the well-known Sobolev embedding Theorem and of Rellich's lemma in the case $p = 2$. Let n denote the dimension of M . We define the *scaling weight* of $L_k^p(M, E)$ by letting

$$w(k, p) := k - \frac{n}{p}.$$

Theorem (A, 1.1) (Sobolev embedding). (cf. [4], Thm. 2.20).

Consider a closed, oriented manifold M and a vector bundle $E \rightarrow M$.

- (i) *Suppose $k_1 \geq k_2$ and $w(k_1, p_1) \geq w(k_2, p_2)$. Then there is a bounded inclusion $L_{k_1}^{p_1}(M, E) \subset L_{k_2}^{p_2}(M, E)$.*
- (ii) *Suppose $w(k, p) > r \in \mathbb{N}$. Then every L_k^p -section of E can be represented by a C^r -section. Moreover, the inclusion $L_k^p(M, E) \subset C^r(E)$ is bounded with respect to the norm on $C^r(E)$ given by uniform convergence of the involved derivatives.*

Theorem (A, 1.2) (Rellich-Kondrachov). (cf. [4], Thm. 2.34).

Consider a closed, oriented manifold M and a vector bundle $E \rightarrow M$. If $k_1 > k_2$ and $w(k_1, p_1) > w(k_2, p_2)$, then the inclusion map $L_{k_1}^{p_1}(M, E) \subset L_{k_2}^{p_2}(M, E)$ is a compact operator. Moreover, the embedding in part (ii) is always compact.

The Sobolev embedding Theorem combined with the well-known Hölder inequality leads to multiplication theorems for the Sobolev spaces L_k^p .

Proposition (A, 1.3) (Sobolev multiplication). *Consider a closed, oriented manifold M and vector bundles $E_1, E_2, F \rightarrow M$, endowed with a bilinear bundle map $b : E_1 \oplus E_2 \rightarrow F$. Let $k_1, k_2, l \in \mathbb{N}$ with $k_1, k_2 \geq l$, and $p_1, p_2, q \in (1, \infty)$ such that $p_1, p_2 \geq q$.² If $w(k_1, p_1) + w(k_2, p_2) > w(l, q)$, then b extends to a bounded bilinear map*

$$b : L_{k_1}^{p_1}(M, E_1) \times L_{k_2}^{p_2}(M, E_2) \rightarrow L_l^q(M, F).$$

Remark. Although this theorem is frequently used in the analysis of non-linear partial differential equations, it is difficult to find a reference in the literature. In a slightly different form, the theorem can be found in Palais [48], Ch. 9. To be self-contained, we shall present a proof.

Proof. Step 1: The Hölder inequality revisited:

We recall that for any $p'_1, p'_2, q' \in (1, \infty)$ such that $p'_1, p'_2 > q'$, the Hölder inequality implies that there is a continuous multiplication $L^{p'_1} \times L^{p'_2} \rightarrow$

²Observe that this implies that $w(k_i, p_i) \geq w(l, q)$.

$L^{q'}$ provided that $\frac{1}{p'_1} + \frac{1}{p'_2} = \frac{1}{q'}$. Moreover, on compact manifolds there is a continuous embedding of $L^{q'}$ in L^q whenever $q' \geq q$. Hence, under the assumption that $\frac{1}{p'_1} + \frac{1}{p'_2} \leq \frac{1}{q}$ or, equivalently, if $w(0, p'_1) + w(0, p'_2) \geq w(0, q)$, we obtain a continuous multiplication $L^{p'_1} \times L^{p'_2} \rightarrow L^q$.

Step 2: The case $l = 0$:

We have to show that there exists a continuous multiplication

$$L_{k_1}^{p_1} \times L_{k_2}^{p_2} \rightarrow L^q. \quad (*)$$

For this we have to study different cases:

Case 1: There exists $i \in \{1, 2\}$ such that $w(k_i, p_i) > 0$. Without loss of generality, we may suppose that $i = 1$. Then $L_{k_1}^{p_1}$ embeds continuously in C^0 . Furthermore, we have an inclusion of $L_{k_2}^{p_2}$ in L^{p_2} . Therefore, there is a bounded map

$$L_{k_1}^{p_1} \times L_{k_2}^{p_2} \rightarrow L^{p_2}.$$

Moreover, L^{p_2} embeds continuously in L^q for $p_2 \geq q$. This proves $(*)$.

Case 2: $w(k_i, p_i) \leq 0$ for $i = 1, 2$. Since $w(k_1, p_1) + w(k_2, p_2) > w(0, q)$, this implies that $w(k_i, p_i) > w(0, q)$ for $i = 1, 2$. Hence, we can find $p'_i \in (q, \infty)$ such that

$$w(k_i, p_i) \geq w(0, p'_i) > w(0, q) \quad \text{and} \quad w(0, p'_1) + w(0, p'_2) \geq w(0, q).$$

Applying the considerations of *Step 1*, we deduce that there exists a continuous multiplication

$$L^{p'_1} \times L^{p'_2} \rightarrow L^q.$$

On the other hand, there are bounded inclusions $L_{k_i}^{p_i} \subset L^{p'_i}$ which yields $(*)$ in the case at hand.

Step 3: The general case:

Let us now assume that l is chosen arbitrarily. Fixing $m \leq l$, we deduce that for every section $u \in L_{k_1}^{p_1}(M, E_1)$ and every $v \in L_{k_2}^{p_2}(M, E_2)$,

$$\begin{aligned} |\nabla^m(b(u, v))| &\leq \text{const} \cdot \left| \sum_{\substack{k+i+j \\ =m}} (\nabla^k b)(\nabla^i u, \nabla^j v) \right| \\ &\leq \text{const} \cdot \sum_{\substack{k+i+j \\ =m}} |(\nabla^k b)(\nabla^i u, \nabla^j v)| \leq \text{const} \cdot \sum_{i+j \leq m} |\nabla^i u| \cdot |\nabla^j v|. \end{aligned}$$

Observe that $\nabla^i u \in L_{k_1-i}^{p_1}$ and $\nabla^j v \in L_{k_2-j}^{p_2}$. Whenever $i + j \leq m$,

$$w(k_1 - i, p_1) + w(k_2 - j, p_2) > w(l - m, q) \geq w(0, q).$$

Hence, $\nabla^i u$ and $\nabla^j v$ satisfy the conditions of *Step 2*. Therefore,

$$\begin{aligned} \|\nabla^m(b(u, v))\|_{L^q} &\leq \text{const} \cdot \sum_{i+j \leq m} \|\nabla^i u\|_{L_{k_1-i}^{p_1}} \cdot \|\nabla^j v\|_{L_{k_2-j}^{p_2}} \\ &\leq \text{const} \cdot \sum_{i+j \leq m} \|\nabla^i u\|_{L_{k_1-i}^{p_1}} \cdot \|\nabla^j v\|_{L_{k_2-j}^{p_2}} \\ &\leq \text{const} \cdot \sum_{i+j \leq m} \|u\|_{L_{k_1}^{p_1}} \cdot \|v\|_{L_{k_2}^{p_2}} \leq \text{const} \cdot \|u\|_{L_{k_1}^{p_1}} \cdot \|v\|_{L_{k_2}^{p_2}} \end{aligned}$$

so that $\|b(u, v)\|_{L_l^q} \leq \text{const} \cdot \|u\|_{L_{k_1}^{p_1}} \cdot \|v\|_{L_{k_2}^{p_2}}$. This proves the assertion. \square

Example (A, 1.4). Let M be a compact and oriented Riemannian 3-manifold, and let $b : E_1 \otimes E_2 \rightarrow F$ a bilinear bundle morphism between arbitrary vector bundles over M . The above proposition shows that b induces a bounded bilinear map

$$b : L_k^2(M, E_1) \times L_k^2(M, E_2) \longrightarrow L_k^2(M, F) \quad (\text{A, 1})$$

whenever k satisfies $2k - \frac{6}{2} > k - \frac{3}{2}$, that is, whenever $k \geq 2$.

In particular, if E is a bundle of algebras, then the Hilbert spaces $L_k^2(M, E)$ are Banach algebras provided that $k \geq 2$. This observation yields natural choices for the involved Sobolev orders when we want to equip a group of gauge transformations with a Sobolev structure. Furthermore, if the bundle of algebras acts on some vector bundle F , then the proposition shows that for an appropriate choice of l , the Banach algebra $L_k^2(M, E)$ acts continuously on $L_l^2(M, F)$. For example, we have an associated continuous multiplication

$$b : L_2^2(M, E) \times L_1^2(M, F) \rightarrow L_1^2(M, F),$$

i.e., $L_1^2(M, F)$ is an $L_2^2(M, E)$ -module. This fact plays an important role when the action of the group of gauge transformations on the space of sections of a vector bundle is modelled in the context of Sobolev spaces.

2 Analytic properties of elliptic operators

Let E and F denote Euclidean or Hermitian vector bundles over a compact and oriented Riemannian manifold M . Furthermore, suppose that

$P : C^\infty(M, E) \rightarrow C^\infty(M, F)$ is a differential operator of order m , i.e., P is expressed in local coordinates as

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_\alpha},$$

where a_α are smooth matrix valued functions, while α is a multi index. Then the *principal symbol* of P is locally defined by

$$\sigma_m(P)_{(x,\xi)} = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha.$$

It gives rise to a bundle map $\sigma_m(P) : \pi^*E \rightarrow \pi^*F$, where $\pi : T^*M \rightarrow M$ denotes the bundle projection of the cotangent bundle.

Associated to every differential operator P there is the so-called *formal adjoint*³ $P^t : C^\infty(M, F) \rightarrow C^\infty(M, E)$. It is defined by the property that for every $u \in C^\infty(M, E)$ and every $v \in C^\infty(M, F)$,

$$\int_M \langle Pu, v \rangle_F dv_g = \int_M \langle u, P^t v \rangle_E dv_g.$$

Deriving an explicit formula via integration by parts yields that the formal adjoint always exists and is again a differential operator of order m . Moreover, it is uniquely characterized by the above property. The principal symbols of P and P^t are related by

$$\sigma_m(P^t) = (-1)^m \sigma_m(P)^*,$$

where $\sigma_m(P)^*$ denotes the adjoint of the bundle map $\sigma_m(P)$ with respect to the induced metrics.

Unbounded operators. Working in the context of Sobolev spaces is readily appreciated by the elementary fact that for each $k \in \mathbb{N}$ and $p \in [1, \infty)$, a differential operator P induces bounded linear maps

$$P_{k,p} : L_{k+m}^p(M, E) \rightarrow L_k^p(M, F)$$

and

$$P_{k,p}^t = (P^t)_{k,p} : L_{k+m}^p(M, F) \rightarrow L_k^p(M, E).$$

³In contrast to the main part of this text, we now denote the formal adjoint with the superscript t instead of $*$. This is because we want to distinguish it clearly from the functional analytic adjoint (see below).

Investigating the functional analytic properties of these maps leads naturally to the theory of unbounded operators in Banach spaces (cf. Kato [23], Sec. III.5 and Sec. V.3). We briefly want to fix some notation in this context. For notational convenience we will drop the reference to p and simply write P_k , the value of p being understood from the context.

As a consequence of the above observation, P induces an unbounded operator $P_0 : L^p(M, E) \supset L_m^p(M, E) \rightarrow L^p(M, F)$ with a dense domain $\text{dom}(P_0) := L_m^p(M, E)$. Mutatis mutandis, the same holds for P^t . If $p = 2$, there is no reason to expect, though, that the operator P_0^t coincides with the *functional analytic* adjoint $(P_0)^*$. Recall that

$$v \in \text{dom}(P_0)^* : \Longleftrightarrow \exists_{w \in L^2(M, E)} \forall_{u \in \text{dom}(P_0)} : \int_M \langle Pu, v \rangle_F dv_g = \int_M \langle u, w \rangle_E dv_g,$$

and $(P_0)^*v := w$. The operator $(P_0)^*$ is linear and densely defined because $C^\infty(M, F) \subset \text{dom}(P_0)^*$. If $E = F$ and $P_0 = (P_0)^*$ as unbounded operators in Hilbert space, then P_0 is called *self-adjoint*. If we only have $P = P^t$, then we call P *formally self-adjoint*.

Remark. Note that the notion of the *functional analytic* adjoint is only well-defined if $p = 2$ because only then L^p is a Hilbert space. In contrast to that, the *formal* adjoint of a differential operator makes perfect sense in the L^p context for all p since it is a differential operator in its own right.

Elliptic operators. The unbounded operators induced by P have surprising functional analytic properties if P is *elliptic*, i.e., if its principal symbol $\sigma_m(P) : \pi^*E \rightarrow \pi^*F$ is an isomorphism off the zero section. We note that this implies that E and F have the same rank. Due to the relation of their principal symbols, ellipticity of P implies that P^t is elliptic as well. The following theorem lies at the heart of the theory of elliptic operators.

Theorem (A, 2.1). *Let $P : C^\infty(M, E) \rightarrow C^\infty(M, F)$ be an elliptic operator. Then the following holds.*

(i) “*Elliptic estimate*”: For $u \in L_{k+m}^p(M, E)$,

$$\|u\|_{L_{k+m}^p} \leq \text{const} \cdot (\|Pu\|_{L_k^p} + \|u\|_{L_k^p}), \quad (\text{A, 2})$$

where $k \in \mathbb{N}$ and $p \in [1, \infty)$.

(ii) “*Elliptic regularity*”: If $u \in L^p(M, E)$ satisfies $Pu \in L_k^p(M, F)$ weakly, i.e., if there exists $v \in L_k^p(M, F)$ such that

$$\forall_{w \in C^\infty(M, F)} : \int_M \langle u, P^t w \rangle_E dv_g = \int_M \langle v, w \rangle_F dv_g,$$

then $u \in L_{k+m}^p(M, E)$.

In particular, if $u \in L_{k+m}^p(M, E)$ satisfies $Pu = 0$, then u is smooth. Hence, the kernels of the operators P_k coincide and consist solely of smooth sections.

The proof of the elliptic estimate is quite technical and can be found in many textbooks (cf. Aubin [4], Sec. 3.6 and references therein, or Gilbarg & Trudinger [18]). Elliptic regularity follows from the elliptic estimate via smoothing arguments and can also be found in the textbooks cited above.

This theorem together with well-known results from functional analysis are the key to an understanding of the analytic properties of elliptic operators. We now start including some proofs since our discussion in the main part of this thesis relies on a detailed understanding of the next results.

Proposition (A, 2.2). *For any $k \in \mathbb{N}$, P_k is a closed unbounded operator $L_k^p(M, E) \rightarrow L_k^p(M, F)$ with domain $L_{k+m}^p(M, E)$.*

Proof. We have to show that the graph of P_k , which is given by

$$\{(u, P_k u) \mid u \in \text{dom}(P_k)\} \subset L_k^p(M, E) \times L_k^p(M, F),$$

is a closed subspace. Suppose that (u_n) is a sequence in $\text{dom}(P_k)$, i.e., in $L_{k+m}^p(M, E)$ such that $u_n \rightarrow u$ in $L_k^p(M, E)$ and $Pu_n \rightarrow v$ in $L_k^p(M, F)$. We claim that $u \in L_{k+m}^p(M, E)$ and $Pu = v$. The elliptic estimate

$$\|u_n - u_{n'}\|_{L_{k+m}^p} \leq \text{const} \cdot (\|Pu_n - Pu_{n'}\|_{L_k^p} + \|u_n - u_{n'}\|_{L_k^p})$$

shows that (u_n) is a Cauchy sequence in $L_{k+m}^p(M, E)$ thus converging in $L_{k+m}^p(M, E)$ to (the same limit point) u . By continuity of

$$P : L_{k+m}^p(M, E) \rightarrow L_k^p(M, F),$$

we infer that $Pu = v$. □

Proposition (A, 2.3). *If $p = 2$, P and its formal adjoint P^t satisfy $(P_0)^* = P_0^t$ and $(P_0^t)^* = P_0$.*

Proof. It suffices to show that $\text{dom}(P_0)^* = \text{dom}(P_0^t) (= L_m^2(M, F))$. Suppose that $v \in \text{dom}(P_0)^*$. Then there exists $w \in L^2(M, E)$ such that for any $u \in \text{dom}(P_0)$,

$$\int_M \langle Pu, v \rangle_F dv_g = \int_M \langle u, w \rangle_E dv_g.$$

Since $C^\infty(M, E) \subset \text{dom}(P_0)$, this implies that $w = P^t v$ weakly in $L^2(M, E)$. By elliptic regularity of P^t , we conclude that $v \in L_m^2(M, F)$. Hence, $(P_0)^* \subset P_0^t$. The other inclusion is obvious. As P^t is also elliptic, the second equality is proved in the same way. □

In addition to the elliptic estimate, we shall need the following result.

Proposition (A, 2.4) (Poincaré inequality). *Let $p \geq 2$, and*

$$u \in (\ker P)^\perp \cap \operatorname{dom}(P_k),$$

where we take the orthogonal complement in $L^2(M, E)$. Then

$$\|u\|_{L_{k+m}^p} \leq \operatorname{const} \cdot \|Pu\|_{L_k^p}, \quad (\text{A, 3})$$

Proof. Invoking the elliptic estimate, we have to show that

$$\|u\|_{L_k^p} \leq \operatorname{const} \cdot \|Pu\|_{L_k^p}.$$

Arguing by contradiction, we assume that there exists a sequence (u_n) in $L_{k+m}^p(M, E)$ such that all u_n are L^2 -orthogonal to $\ker P$ and

$$\|u_n\|_{L_k^p} > n \cdot \|Pu_n\|_{L_k^p}.$$

Without loss of generality we may also demand that $\|u_n\|_{L_k^p} = 1$. Then the last inequality shows that $\|Pu_n\|_{L_k^p} \rightarrow 0$, and the elliptic estimate imposes an L_{k+m}^p -bound on (u_n) . We deduce from the Rellich-Kondrachov Theorem that a subsequence of (u_n) converges in $L_k^p(M, E)$ to, say, u . In particular, $\|u\|_{L_k^p} = 1$. As $Pu_n \rightarrow 0$ and P_k is closed, we infer that $u \in L_{k+m}^p(M, E)$ and $Pu = 0$. On the other hand, all u_n are L^2 -orthogonal to $\ker P$ which yields that the same holds for u . Hence, $u \in \ker P \cap (\ker P)^\perp$ so that necessarily $u = 0$. This contradicts $\|u\|_{L_k^p} = 1$. \square

As an application of this result, we now deduce the Fredholm property of elliptic operators.

Theorem (A, 2.5). *Let M be a closed, oriented manifold and let $P : C^\infty(M, E) \rightarrow C^\infty(M, F)$ be an elliptic differential operator. Then the following holds.*

- (i) *For every $k \in \mathbb{N}$ and $p \geq 2$, the operators P_k and P_k^t are semi-Fredholm, i.e., they have closed ranges and finite dimensional kernels.*
- (ii) *“Hodge decomposition”: There is an L^2 -orthogonal decomposition*

$$L_k^p(M, F) = \operatorname{im} P_k \oplus \ker(P^t). \quad (\text{A, 4})$$

In particular, P_k is Fredholm, i.e., its kernel and cokernel are finite dimensional. Moreover, the Fredholm index

$$\operatorname{ind} P_k := \dim(\ker P_k) - \dim(\operatorname{coker} P_k)$$

neither depends on k nor on p .

Proof. Let (v_n) be a sequence in the image of P_k which converges in $L_k^p(M, F)$ and let v denote the limit point. We choose a sequence (u_n) in $L_{k+m}^p(M, E)$ such that $Pu_n = v_n$. Since $\ker P \subset C^\infty(M, E)$ and $L_k^p \subset L^2$ (because $p \geq 2$), we may assume that all u_n are L^2 -orthogonal to $\ker P$. We then infer from the Poincaré inequality (A, 3) that (u_n) is a Cauchy sequence in L_{k+m}^p hence converging to, say, $u \in L_{k+m}^p(M, E)$. Since $P_k : L_{k+m}^p(M, E) \rightarrow L_k^p(M, F)$ is continuous, $Pu = v$. Therefore, $\text{im } P_k$ is closed in $L_k^p(M, F)$.

To prove the finite dimensionality of $\ker P$, we want to use the well-known fact that a Banach space is finite dimensional if and only if the unit sphere is sequentially compact. Hence, suppose (u_n) is a sequence in $\ker P$ with $\|u_n\|_{L_{k+m}^p} = 1$. As a consequence of the Rellich-Kondrachov Theorem, (u_n) contains a subsequence which converges in $L_k^p(M, E)$ to some limit point u . Since $Pu_n = 0$, the elliptic estimate yields that this subsequence is also a Cauchy sequence in $L_{k+m}^2(M, E)$ thus also converging to u with respect to the L_{k+m}^2 -topology. Continuity of P_k implies that $Pu = 0$ so that $u \in \ker P$. Clearly, $\|u\|_{L_{k+m}^p} = 1$ and this shows that the unit sphere in $\ker P$ is sequentially compact. In the same way, replacing P with P^t , we get that P^t is a semi-Fredholm operator. Thus we have proved (i).

Concerning (ii), let us first content ourselves to the case $k = 0$ and $p = 2$. Since P_0 has a closed range in $L^2(M, F)$, we have an L^2 -orthogonal decomposition

$$L^2(M, F) = \text{im } P_0 \oplus \ker(P_0)^* = \text{im } P_0 \oplus \ker P_0^t.$$

Here, we have used that $(P_0)^*$ and P_0^t coincide. Since P^t is elliptic, $\ker P_0^t = \ker P^t$. Hence, the assertion is proved in the case at hand.

To obtain the general case, we intersect the above decomposition with $L_k^p(M, F)$. Since $L_k^p \subset L^2$ and $\ker P^t \in C^\infty(M, F)$, this yields an L^2 -orthogonal decomposition

$$L_k^p(M, F) = (\text{im } P_0 \cap L_k^p(M, F)) \oplus \ker P^t.$$

Elliptic regularity of P implies that $\text{im } P_0 \cap L_k^p(M, F) = \text{im } P_k$ which establishes (ii).

From the Hodge decomposition we deduce that $\text{coker } P_k \cong \ker P^t$ where the right hand side is finite dimensional and independent of k and p . Moreover, we have already observed that $\ker P$ is finite dimensional and neither depends on k nor p . This implies the assertion about the independence of the Fredholm index. \square

Discrete spectrum. Let $P : C^\infty(M, E) \rightarrow C^\infty(M, E)$ be a formally self-adjoint, elliptic differential operator. Then, according to Proposition (A, 2.3),

the associated operator P_0 in $L^2(M, E)$ is self-adjoint. This implies that its spectrum,

$$\text{spec}(P_0) := \{ \lambda \in \mathbb{C} \mid P_0 - \lambda : L_m^2(M, E) \rightarrow L^2(M, E) \text{ is not invertible} \},$$

is a (closed) subset of \mathbb{R} . As a consequence of the Rellich Lemma, any element $\lambda \in \mathbb{C} \setminus \text{spec}(P_0)$ defines a compact operator $(P_0 - \lambda)^{-1} : L^2(M, E) \rightarrow L_m^2(M, E) \subset L^2(M, E)$. Therefore, P_0 is said to have *compact resolvent*. From the spectral theory of compact operators, it is easy to deduce that the spectrum of a self-adjoint operator having compact resolvent consists solely of discrete eigenvalues of finite multiplicity, which form an unbounded subset of \mathbb{R} (cf. Kato [23], Thm. III.6.29). One sometimes refers to this property by calling P_0 an operator with *discrete spectrum*. In addition to that, elliptic regularity of P_0 implies that the corresponding eigenvectors are smooth.

Injectively elliptic operators. The requirement that $\sigma_m(P)$ is an isomorphism naturally splits into two parts, namely injectivity and surjectivity of the symbol. A differential operator $P : C^\infty(M, E) \rightarrow C^\infty(M, F)$ of order m is called *injectively elliptic* if its principal symbol is injective off the zero section. The elliptic estimate (A, 2) also holds for differential operators of this kind:

Theorem (A, 2.6). *Let $P : C^\infty(M, E) \rightarrow C^\infty(M, F)$ be an injectively elliptic operator of order m . Then the following holds.*

(i) “*Elliptic estimate*”: For all $k \in \mathbb{N}$ and $u \in L_{k+m}^2(M, E)$,

$$\|u\|_{L_{k+m}^2} \leq \text{const} \cdot (\|Pu\|_{L_k^2} + \|u\|_{L_k^2}).$$

(ii) “*Elliptic regularity*”: If $u \in L^2(M, E)$ satisfies $Pu \in L_k^2(M, F)$ weakly, then $u \in L_{k+m}^2(M, E)$.

Sketch of proof. We shall only give a proof in the case $k \geq m$, since then the assertions are immediate consequences of Theorem (A, 2.1) applied to the elliptic operator

$$P^t P : C^\infty(M, E) \rightarrow C^\infty(M, E)$$

of order $2m$. Since $P_{k-m}^t : L_k^2(M, F) \rightarrow L_{k-m}^2(M, E)$ is bounded, the elliptic estimate for $P^t P$,

$$\|u\|_{L_{k+m}^2} \leq \text{const} \cdot (\|P^t Pu\|_{L_{k-m}^2} + \|u\|_{L_{k-m}^2}),$$

implies that also

$$\|u\|_{L_{k+m}^2} \leq \text{const} \cdot (\|Pu\|_{L_k^2} + \|u\|_{L_{k-m}^2}).$$

Since $\|\cdot\|_{L_{k-m}^2} \leq \|\cdot\|_{L_k^2}$, we obtain the elliptic estimate for P and thus (i). Similarly, we get (ii) in the following way: Let $u \in L^2(M, E)$ and $Pu = v$ weakly with $v \in L_k^2(M, F)$, $k \geq m$. Then it is immediate that

$$P^t Pu = P^t v \in L_{k-m}^2(M, E)$$

weakly. Therefore, $u \in L_{k-m+2m}^2 = L_{k+m}^2$. The proof in the general case has to be carried out by introducing Sobolev spaces of negative order as the dual spaces—endowed with the operator norm—to the corresponding spaces of positive order. The elliptic estimate continues to hold for these spaces, and the above arguments carry over to this setting. \square

Remark. Although we have only given a rigid proof of the above theorem in the case $k \geq m$, we shall use it for all $k \geq 0$. Yet, we have formulated the theorem only for $p = 2$ since otherwise, the above proof does not easily carry over to the case $k < m$.

One now establishes functional analytic properties much as before, taking care, however, that all arguments involving ellipticity of P 's formal adjoint are no longer valid in the context of injectively elliptic operators. The following Proposition summarizes the results we need.

Proposition (A, 2.7). *Let $P : C^\infty(M, E) \rightarrow C^\infty(M, F)$ be an injectively elliptic operator of order m and let $k \in \mathbb{N}$. Then*

- (i) *The unbounded operator P_k is a closed unbounded semi-Fredholm operator*

$$P_k : L_k^2(M, E) \supset L_{k+m}^2(M, E) \rightarrow L_k^2(M, F)$$

with finite dimensional kernel.

- (ii) *The formal adjoint of P satisfies $(P_0^t)^* = P_0$.*

- (iii) *There is an L^2 -orthogonal decomposition*

$$L_m^2(M, F) = \text{im } P_m \oplus \ker P_0^t. \quad (\text{A, 5})$$

Proof. Part (i) and (ii) are proved exactly as before. For this note that the Poincaré inequality (A, 3) also holds for injectively elliptic operators. Regarding (iii), we first have an L^2 -orthogonal decomposition

$$L^2(M, F) = \text{im } P_0 \oplus \ker(P_0)^*, \quad (\text{A, 6})$$

for which we invoke that P_0 has closed range. Suppose $v \in L_m^2(M, F)$. As an element of $L^2(M, F)$, we may decompose v according to the above

as $v = Pu + w$, with $u \in L_m^2(M, E)$ and $w \in \ker P_0^*$. Applying P^t to this equation, we deduce that $P^t v = P^t Pu$ weakly in $L^2(M, E)$. Elliptic regularity of $P^t P$ then guarantees that $u \in L_{2m}^2(M, E)$ so that in particular, $w = v - Pu \in L_m^2(M, F)$. As $\ker P_0^* \cap L_m^2(M, F) = \ker P_0^t$, the assertion follows. \square

In contrast to the elliptic case, $\ker(P_0)^*$ is in general neither finite dimensional nor does it coincide with the kernel of the formal adjoint. However, the above proposition implies

Corollary (A, 2.8). *Let $P : C^\infty(M, E) \rightarrow C^\infty(M, F)$ be an injectively elliptic operator of order m . Then*

$$\operatorname{im} P_0 = \overline{\operatorname{im} P_m}^{L^2} \quad \text{and} \quad \ker(P_0)^* = \overline{\ker P_0^t}^{L^2}.$$

Proof. Since $\operatorname{im} P_m \subset \operatorname{im} P_0$ and $\ker P_0^t \subset \ker(P_0)^*$,

$$\overline{\operatorname{im} P_m}^{L^2} \subset \operatorname{im} P_0 \quad \text{and} \quad \overline{\ker P_0^t}^{L^2} \subset \ker(P_0)^* \quad (\text{A, 7})$$

because the subspaces $\operatorname{im} P_0$ and $\ker(P_0)^*$ are closed in $L^2(M, F)$. As the subspace $L_m^2(M, F)$ is dense in $L^2(M, F)$, we deduce from (A, 6) and (A, 5) that

$$\operatorname{im} P_0 \oplus \ker(P_0)^* = \overline{\operatorname{im} P_m}^{L^2} \oplus \overline{\ker P_0^t}^{L^2}$$

as L^2 -orthogonal decompositions. Together with (A, 7), this implies the assertion. \square

Non-smooth coefficients. In gauge theory one usually works with nonlinear partial differential equations. In view of the implicit function theorem, it is promising to model the nonlinear partial differential operator on a suitable Sobolev space and study its differential in order to gather information about the set of solutions. This linearization fits into the context of linear differential operators we have described above, though, with a slight modification: The point in the Sobolev space, where we are linearizing the operator enters the differential and we usually obtain a differential operator with Sobolev coefficients. However, the actual generalization we have to make is only a minor one. The underlying observation is that in the situation we shall encounter, the nonlinear part of the partial differential equation is of order 0. This implies that linearizing the equation at different points yields operators differing only by a compact operator. It is then a well-established fact that the functional analytic properties of the two linear operators are essentially the same.

Theorem (A, 2.9). *Let $P : C^\infty(M, E) \rightarrow C^\infty(M, F)$ be an injective elliptic operator of order m . Moreover, let $T : L_m^2(M, E) \rightarrow L^2(M, E)$ be a compact⁴ operator and consider $P + T$ with $\text{dom}(P + T) = L_m^2(M, E)$. Then the following holds*

- (i) *The unbounded operator $P + T$ is closed and semi-Fredholm operator with finite dimensional kernel.*
- (ii) *$L^2(M, F) = \text{im}(P + T) \oplus \ker(P + T)^*$.*
- (iii) *Suppose that $F = E$ and that P is formally self-adjoint, and T is symmetric with respect to the L^2 scalar product. Then $P + T$ is a self-adjoint Fredholm operator in $L^2(M, E)$ with compact resolvent.*

Part (i) and (ii) follow from Proposition (A, 2.7) and Theorem V.5.26 of [23] about relatively compact perturbations of semi-Fredholm operators. Part (iii) is an immediate consequence of Theorem 9.9 in [58], see also Sec. V.4 in [23].

⁴In this situation, T is called *relatively compact* with respect to P .

Appendix B

The determinant line bundle

In this appendix we present a version of how to construct a canonical line bundle over the space of Fredholm operators. Since we shall not need this notion in the context of Banach spaces, we restrict ourselves immediately to Hilbert spaces although the situation is more or less the same. For a discussion of different possibilities to construct the determinant bundle, we refer to [6], Ch. 3.

Let $\mathcal{F}(H_1, H_2)$ denote the set of bounded Fredholm operators between two separable \mathbb{K} -Hilbert spaces H_1 and H_2 .

Definition (B, 1.10). The *determinant line* of $T \in \mathcal{F}(H_1, H_2)$ is the vector space

$$\det T := \det(\ker T) \otimes (\det(\operatorname{coker} T))^*,$$

where $(\dots)^*$ denotes the dual space. Recall that for each n -dimensional \mathbb{K} -vector space V , the space $\det(V)$ is defined as the top exterior power $\Lambda^n V$. In particular, $\det\{0\}$ is the underlying scalar field \mathbb{K} .

The space of Fredholm operators. We equip $\mathcal{F}(H_1, H_2)$ with the topology induced by the operator norm on $\mathcal{L}(H_1, H_2)$, the latter denoting the Banach space of bounded linear maps. Let $\mathcal{U} \subset \mathcal{F}(H_1, H_2)$ be a connected open subset. It is well-known that the map $T \mapsto \operatorname{ind}(T)$ is constant on \mathcal{U} . Assume for a moment that the assignment $T \mapsto \dim(\ker T)$ is also constant on \mathcal{U} . Under this assumption, defining $(\operatorname{Ker})_T := \ker T$ and $(\operatorname{Coker})_T := \operatorname{coker} T$ for every $T \in \mathcal{U}$ yields vector bundles $\operatorname{Ker} \rightarrow \mathcal{U}$ and $\operatorname{Coker} \rightarrow \mathcal{U}$. We can thus form the line bundle

$$\operatorname{Det} := \det(\operatorname{Ker}) \otimes (\det(\operatorname{Coker}))^* \longrightarrow \mathcal{U}.$$

However, there is no immediate way of endowing the collection $\bigcup_T \det T$ with the structure of a line bundle if $\dim(\ker T)$ varies with T . To achieve this, we introduce the following concept.

Definition (B, 1.11). Let $\mathcal{U} \subset \mathcal{F}(H_1, H_2)$ and let $K : \mathcal{U} \rightarrow \mathcal{L}(V, H_2)$ be a continuous map, where V is a finite dimensional Hilbert space. If the operator

$$T_K : H_1 \oplus V \rightarrow H_2, \quad (e, v) \mapsto Te + K(T)v,$$

is surjective for every $T \in \mathcal{U}$, we call K a *stabilizer over \mathcal{U}* .

Let $K : \mathcal{U} \rightarrow \mathcal{L}(V, H_2)$ be a stabilizer over some open subset $\mathcal{U} \subset \mathcal{F}_n$, the latter denoting the component of Fredholm operators of index n . Then for every $T \in \mathcal{U}$, the operator T_K is surjective and Fredholm with index equal to $n + \dim V$. Therefore, the collection $\bigcup_{T \in \mathcal{U}} \ker T_K$ forms a well-defined vector bundle which we denote by $\text{Ker}_K \rightarrow \mathcal{U}$. We can then form the line bundle

$$\det(\text{Ker}_K) \longrightarrow \mathcal{U}.$$

We shall see in Proposition (B, 1.17) below, that there exists a natural isomorphism

$$\det T \cong \det(\ker T_K) \otimes (\det V)^* \tag{B, 1}$$

for any $T \in \mathcal{U}$. Thus, the idea is to define the structure of a line bundle on $\bigcup_{T \in \mathcal{U}} \det T$ via the above isomorphisms. Before doing so, let us first recall the well-known fact that stabilizers exist in abundance.

Lemma (B, 1.12). *For each $T_0 \in \mathcal{F}(H_1, H_2)$, there exist a finite dimensional subspace $V \subset H_2$ and an open neighbourhood \mathcal{U} of T_0 such that the constant map $K := (V \hookrightarrow H_2)$ is a stabilizer over \mathcal{U} .*

Proof. Let $V := \text{im } T_0^\perp$. Since T_0 is Fredholm, V is a finite dimensional subspace of H_2 . We define $W := (\ker T_0)^\perp \subset H_1$ and $K := (V \hookrightarrow H_2)$. Then the map

$$S : \mathcal{F}(H_1, H_2) \rightarrow \mathcal{L}(W \oplus V, H_2), \quad S(T) := T_K|_{W \oplus V}$$

is continuous. Moreover, the construction is made in such a way that $S(T_0)$ is invertible. It is well-known that the set of invertible elements is open in $\mathcal{L}(W \oplus V, H_2)$. Hence, there exists an open neighbourhood \mathcal{U} of T_0 such that for every $T \in \mathcal{U}$, the operator $S(T) : W \oplus V \rightarrow H_2$ is invertible. In particular, the operator $T_K : H_1 \oplus V \rightarrow H_2$ is surjective. \square

The determinant line of a Fredholm operator. Our next aim is to describe the isomorphism (B, 1). This requires some linear algebra so that we shall restrict to the case of a single Fredholm operator $T : H_1 \rightarrow H_2$ in order to simplify notation.

Lemma (B, 1.13). *Let $K : V \rightarrow H_2$ be a stabilizer of $T \in \mathcal{F}(H_1, H_2)$, and let $P_V := \text{Proj}_V : H_1 \oplus V \rightarrow V$, and $F := \text{Proj}_{\text{coker } T} \circ K$, where Proj denotes the orthogonal projection. Then the sequence*

$$0 \longrightarrow \ker T \longrightarrow \ker T_K \xrightarrow{P_V} V \xrightarrow{F} \text{coker } T \longrightarrow 0 \quad (\text{B, 2})$$

is exact. Hence, there exists a natural isomorphism

$$\det(\ker T) \otimes \det V \cong \det(\ker T_K) \otimes \det(\text{coker } T). \quad (\text{B, 3})$$

Proof. The existence of the isomorphism (B, 3) follows from exactness of (B, 2) and Lemma (B, 1.14) below. Hence, we are left to show exactness of the sequence. Since this is fairly trivial, we only mention that $P_V(\ker T_K) = \ker F$ because

$$v \in P_V(\ker T_K) \iff Kv \in \text{im } T \iff Fv = \text{Proj}_{\text{coker } T} \circ Kv = 0. \quad \square$$

Lemma (B, 1.14). *Let*

$$0 \longrightarrow V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} V_0 \longrightarrow 0 \quad (\text{B, 4})$$

be an exact sequence of finite dimensional vector spaces. Then there exists a natural isomorphism

$$\bigotimes_{2k \leq n} \det V_{n-2k} \cong \bigotimes_{2k \leq n-1} \det V_{n-1-2k} \quad (\text{B, 5})$$

Sketch of proof. Let $\eta_n \otimes \eta_{n-2} \otimes \dots$ be an element of the left-hand side of (B, 5). For each i let $c_i := \dim(V_i / \ker f_i)$. It follows from exactness of (B, 4) that there exist $\omega_i \in \Lambda^{c_i} V_i$ such that

$$\eta_n \otimes \eta_{n-2} \otimes \dots = \omega_n \otimes (f(\omega_{n-1}) \wedge \omega_{n-2}) \otimes \dots$$

Then

$$\omega_n \otimes (f(\omega_{n-1}) \wedge \omega_{n-2}) \otimes \dots \longmapsto (f(\omega_n) \wedge \omega_{n-1}) \otimes (f(\omega_{n-2}) \wedge \omega_{n-3}) \otimes \dots$$

gives the desired isomorphism. It is routine to check that this isomorphism does not depend on the particular choices of $\omega_n, \dots, \omega_0$. \square

Definition (B, 1.15). We call a basis of the form

$$\omega_n \otimes (f(\omega_{n-1}) \wedge \omega_{n-2}) \otimes \dots \in \bigotimes_{2k \leq n} \det V_{n-2k}$$

an *adapted basis* associated to the sequence (B, 4).

Example. Let us illustrate the independence of the adapted basis in the case $n = 2$, i.e., for a short exact sequence

$$0 \longrightarrow V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \longrightarrow 0.$$

Let $\eta_2 \otimes \eta_0 \in \det V_2 \otimes \det V_0$ and choose $\omega_i, \omega'_i \in \Lambda^\bullet V_i$ such that

$$\eta_2 \otimes \eta_0 = \omega_2 \otimes f_1(\omega_1) = \omega'_2 \otimes f_1(\omega'_1).$$

If this expression is nonzero—what we will assume henceforth—there exists $\lambda \in \mathbb{K}^*$ such that $\omega'_2 = \lambda \cdot \omega_2$ and $f_1(\omega'_1) = \lambda^{-1} \cdot f_1(\omega_1)$. Note that this does *not* imply that $\omega'_1 = \lambda^{-1} \cdot \omega_1$ since ω_1 and ω'_1 might span determinant lines of different complements to $\ker f_1$. Yet, writing $\omega_2 = u_1 \wedge \dots \wedge u_{c_2}$, $\omega_1 = v_1 \wedge \dots \wedge v_{c_1}$, and $\omega'_1 = v'_1 \wedge \dots \wedge v'_{c_1}$, we obtain two ordered bases of V_1 ,

$$(f_2(u_1), \dots, f_2(u_{c_2}), v_1, \dots, v_{c_1}) \quad \text{and} \quad (f_2(u_1), \dots, f_2(u_{c_2}), v'_1, \dots, v'_{c_1}).$$

The corresponding transition matrix has the form

$$\begin{pmatrix} 1 & 0 \\ * & A \end{pmatrix},$$

and thus, $f_2(\omega_2) \wedge \omega'_1 = \det(A) \cdot f_2(\omega_2) \wedge \omega_1$. Moreover, since $f_1(\omega'_1) = \lambda^{-1} \cdot f_1(\omega_1)$ and $f_2(u_i) \in \ker f_1$, it follows that $\det A = \lambda^{-1}$. Then

$$\begin{aligned} f_2(\omega'_2) \wedge \omega'_1 &= f_2(\lambda \cdot \omega_2) \wedge \omega'_1 = \lambda \cdot f_2(\omega_2) \wedge \omega'_1 \\ &= \lambda \cdot (\det A) \cdot f_2(\omega_2) \wedge \omega_1 = f_2(\omega_2) \wedge \omega_1. \end{aligned}$$

Sign conventions. Recall that for any 1-dimensional \mathbb{K} -vector spaces L and L' , there are canonical isomorphisms

$$\begin{aligned} L \otimes L' &\cong L' \otimes L, & u \otimes u' &\longmapsto u' \otimes u \\ L^* \otimes L &\cong \mathbb{K}, & u^* \otimes v &\longmapsto u^*[v]. \end{aligned} \tag{B, 6}$$

With these rules it is easy to get (B, 1) from (B, 3). However, there are some subtleties concerning signs. In Appendix C we want to extract a sign,

the so-called *orientation transport*, from the determinant line bundle of a path of self-adjoint Fredholm operators. For this reason we have to be very careful about how to deal with signs. We shall use the Knudsen-Mumford sign conventions [25] which we recall now (see also Nicolaescu [47], Sec. 1.2).

Let V be a finite dimensional \mathbb{K} -vector space. Then the determinant line $\det V$ carries a natural *weight*, namely the natural number $\dim V$. The general concept lying behind this is the following:

Definition (B, 1.16). Let L be a 1-dimensional vector space, and let $w \in \mathbb{Z}$. Then the tuple (L, w) is called a *weighted line*. We define

$$(L, w)^* := (L^*, -w),$$

and, if (L', w') is another weighted line,

$$(L, w) \otimes (L', w') := (L \otimes L', w + w').$$

From now on we consider a determinant line $\det V$ as a weighted line with weight $\dim V$.

In the context of weighted lines (L, w) and (L', w') , the canonical isomorphisms (B, 6) are altered in the following way:

$$\begin{aligned} L \otimes L' &\cong L' \otimes L, & u \otimes u' &\longmapsto (-1)^{ww'} u' \otimes u \\ L^* \otimes L &\cong \mathbb{K}, & u^* \otimes v &\longmapsto (-1)^{\frac{w(w-1)}{2}} u^*[v]. \end{aligned} \tag{B, 7}$$

These are the so-called *Knudsen-Mumford sign conventions*. For the remaining part of this appendix it is understood that we are using (B, 7).

Example. The application to Lemma (B, 1.13) in mind, we shall have a closer look at a four term exact sequence

$$0 \longrightarrow V_3 \xrightarrow{f_3} V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \longrightarrow 0.$$

Then the isomorphism $\det V_3 \otimes \det V_1 \cong \det V_2 \otimes \det V_0$ together with (B, 7) shows that

$$\begin{aligned} \det V_3 \otimes (\det V_0)^* &\cong \det V_3 \otimes \det V_1 \otimes (\det V_1)^* \otimes (\det V_0)^* \\ &\cong \det V_2 \otimes \det V_0 \otimes (\det V_1)^* \otimes (\det V_0)^* \\ &\cong \det V_2 \otimes (\det V_1)^* \otimes \det V_0 \otimes (\det V_0)^* \\ &\cong \det V_2 \otimes (\det V_1)^* \end{aligned}$$

Taking the sign conventions into account, one checks that, in terms of an adapted basis

$$\omega_3 \otimes (f_2(\omega_2) \wedge \omega_1) \in \det V_3 \otimes \det V_1,$$

the induced isomorphism $\det V_3 \otimes (\det V_0)^* \cong \det V_2 \otimes (\det V_1)^*$ is given by

$$\omega_3 \otimes (f_1(\omega_1))^* \longmapsto (-1)^{\frac{(n_0+n_1)(n_0+n_1+1)}{2}} (f_3(\omega_3) \wedge \omega_2) \otimes (f_2(\omega_2) \wedge \omega_1)^*,$$

where $n_i := \dim V_i$, and $(\dots)^*$ denotes the operation of taking the dual.

Translating this example to the situation of Lemma (B, 1.13), we obtain the result we were aiming at in (B, 1):

Proposition (B, 1.17). *Let $T \in \mathcal{F}(H_1, H_2)$ and let $K : V \rightarrow H_2$ be a stabilizer of T . Define $F := \text{Proj}_{\text{coker } T} \circ K$. Then there is a natural isomorphism*

$$\Phi_K : \det(\ker T) \otimes (\det(\text{coker } T))^* \rightarrow \det(\ker T_K) \otimes (\det V)^*.$$

This isomorphism is uniquely defined in the following way. If

$$\xi \otimes (P_V(\eta) \wedge \omega) \in \det(\ker T) \otimes (\det V),$$

is an adapted basis associated to the sequence (B, 2), then

$$\Phi_K(\xi \otimes (F(\omega))^*) = (-1)^{\frac{(n_0+n_1)(n_0+n_1+1)}{2}} (\xi \wedge \eta) \otimes (P_V(\eta) \wedge \omega)^*, \quad (\text{B, 8})$$

where $n_0 := \dim(\text{coker } T)$ and $n_1 := \dim V$.

The advantage of regarding determinant lines as weighted lines together with the above sign conventions is that the isomorphism (B, 5) behaves functorial with respect to morphisms of exact sequences. Instead of going into further detail in the abstract setting, we restrict to the application to determinant lines of Fredholm operators.

Let $K_1 : V_1 \rightarrow H_2$ and $K_2 : V_2 \rightarrow H_2$ be stabilizers of $T \in \mathcal{F}(H_1, H_2)$. Then $K_1 + K_2 : V_1 \oplus V_2 \rightarrow H_2$ is also a stabilizer of T . Hence, Proposition (B, 1.17) yields isomorphisms

$$\Phi_{K_1+K_2} : \det(\ker T) \otimes (\det(\text{coker } T))^* \rightarrow \det(\ker T_{K_1+K_2}) \otimes (\det(V_1 \oplus V_2))^*$$

and, for $i = 1, 2$,

$$\Phi_{K_i} : \det(\ker T) \otimes (\det(\text{coker } T))^* \rightarrow \det(\ker T_{K_i}) \otimes (\det V_i)^*.$$

The next result shows that these isomorphisms are naturally related.

Proposition (B, 1.18). *Let $K_1 : V_1 \rightarrow H_2$ and $K_2 : V_2 \rightarrow H_2$ be stabilizers of $T \in \mathcal{F}(H_1, H_2)$. Then, for $i \in \{1, 2\}$, there exists a natural isomorphism*

$$\Phi_i : \det(\ker T_{K_i}) \otimes (\det V_i)^* \rightarrow \det(\ker T_{K_1+K_2}) \otimes (\det(V_1 \oplus V_2))^*$$

such that the following diagram commutes.

$$\begin{array}{ccc}
 & \det(\ker T_{K_1+K_2}) \otimes \det(V_1 \oplus V_2)^* & \\
 \Phi_1 \nearrow & \uparrow & \nwarrow \Phi_2 \\
 \det(\ker T_{K_1}) \otimes (\det V_1)^* & & \det(\ker T_{K_2}) \otimes (\det V_2)^* \\
 \nwarrow \Phi_{K_1} & \downarrow & \nearrow \Phi_{K_2} \\
 & \det(\ker T) \otimes \det(\operatorname{coker} T)^* &
 \end{array} \tag{B, 9}$$

Proof. We show the assertion for the left triangle of (B, 9). It is easy to check that, with $F_i := \operatorname{Proj}_{\operatorname{coker} T} \circ K_i$,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & 0 & \longrightarrow & V_2 & \xrightarrow{\operatorname{id}} & V_2 & \longrightarrow 0 \\
 & \uparrow & & \uparrow P_{V_2} & & \uparrow P_{V_2} & \uparrow \\
 0 & \longrightarrow & \ker T & \longrightarrow & \ker T_{K_1+K_2} & \xrightarrow{P_{V_1 \oplus V_2}} & V_1 \oplus V_2 \xrightarrow{F_1+F_2} \operatorname{coker} T \longrightarrow 0 \\
 & \uparrow \operatorname{id} & & \uparrow & & \uparrow & \uparrow \operatorname{id} \\
 0 & \longrightarrow & \ker T & \longrightarrow & \ker T_{K_1} & \xrightarrow{P_{V_1}} & V_1 \xrightarrow{F_1} \operatorname{coker} T \longrightarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \uparrow \\
 & 0 & & 0 & & 0 & 0
 \end{array}$$

is a commutative diagram with exact rows and columns. Note that the first vertical short exact sequence stems from the fact that $T_{K_1+K_2}$ is a stabilizer of T_{K_1} . It yields that

$$\det(\ker T_{K_1}) \otimes \det V_2 \cong \det(\ker T_{K_1+K_2}). \tag{B, 10}$$

Moreover, we get from the second vertical short exact sequence that

$$\det V_1 \otimes \det V_2 \cong \det(V_1 \oplus V_2). \tag{B, 11}$$

Now, we deduce from these two isomorphisms that the isomorphism Φ_1 we are looking for is naturally given by the composition

$$\begin{aligned} \det(\ker T_{K_1}) \otimes (\det V_1)^* & \\ & \cong \det(\ker T_{K_1}) \otimes (\det V_1)^* \otimes \det V_2 \otimes (\det V_2)^* \\ & \cong \det(\ker T_{K_1}) \otimes \det V_2 \otimes (\det V_1)^* \otimes (\det V_2)^* \\ & \cong \det(\ker T_{K_1+K_2}) \otimes (\det(V_1 \oplus V_2))^*. \end{aligned}$$

Note that in the last line we have employed that $(L_1 \otimes L_2)^* \cong L_1^* \otimes L_2^*$ for any two weighted lines L_1 and L_2 .

To give an explicit description of Φ_1 , consider adapted bases,

$$\eta_1 \otimes P_{V_2}(\eta_2) \in \det(\ker T_{K_1}) \otimes \det V_2$$

associated to (B, 10), and

$$\omega_1 \otimes P_{V_2}(\omega_2) \in \det V_1 \otimes \det V_2$$

associated to (B, 11). Note that we may take $\omega_2 := P_{V_1 \oplus V_2}(\eta_2)$. This follows from commutativity of the big diagram which particularly shows that

$$P_{V_2}(\omega_2) = P_{V_2} \circ P_{V_1 \oplus V_2}(\eta_2) = P_{V_2}(\eta_2). \quad (\text{B, 12})$$

Using the sign conventions (B, 7), and letting $n_1 := \dim V_1$ and $n_2 := \dim V_2$, one readily checks that Φ_1 is now given by

$$\begin{aligned} \Phi_1(\eta_1 \otimes \omega_1^*) &= (-1)^{n_1 n_2 + \frac{n_2(n_2+1)}{2}} \cdot (\eta_1 \wedge \eta_2) \otimes (\omega_1 \wedge P_{V_2}(\eta_2))^* \\ &= (-1)^{n_1 n_2 + \frac{n_2(n_2+1)}{2}} \cdot (\eta_1 \wedge \eta_2) \otimes (\omega_1 \wedge P_{V_1 \oplus V_2}(\eta_2))^*, \end{aligned} \quad (\text{B, 13})$$

It remains to check that $\Phi_1 \circ \Phi_{K_1} = \Phi_{K_1+K_2}$. To use the explicit description (B, 8) of Φ_{K_1} , we consider an adapted basis

$$\xi \otimes (P_{V_1}(\eta'_1) \wedge \omega'_1) \in \det(\ker T) \otimes (\det V_1)$$

so that

$$\Phi_{K_1}(\xi \otimes F_1(\omega'_1)^*) = (-1)^{\frac{(n_0+n_1)(n_0+n_1+1)}{2}} (\xi \wedge \eta'_1) \otimes (P_{V_1}(\eta'_1) \wedge \omega'_1)^*,$$

where $n_0 := \dim(\text{coker } T)$. Letting $\eta_1 := \xi \wedge \eta'_1$ and $\omega_1 := P_{V_1}(\eta'_1) \wedge \omega'_1$ in (B, 13), we apply Φ_1 to this and get

$$\begin{aligned} \Phi_1 \circ \Phi_{K_1}(\xi \otimes F_1(\omega'_1)^*) &= (-1)^{\frac{(n_0+n_1)(n_0+n_1+1)}{2} + n_1 n_2 + \frac{n_2(n_2+1)}{2}} \\ &\quad \cdot ((\xi \wedge \eta'_1) \wedge \eta_2) \otimes ((P_{V_1}(\eta'_1) \wedge \omega'_1) \wedge P_{V_2}(\eta_2))^*. \end{aligned}$$

To compute $\Phi_{K_1+K_2}(\xi \otimes F(\omega'_1)^*)$, first note that

$$\xi \otimes (P_{V_1 \oplus V_2}(\eta'_1 \wedge \eta_2) \wedge \omega'_1) = \xi \otimes (P_{V_1}(\eta'_1) \wedge P_{V_2}(\eta_2) \wedge \omega'_1)$$

is an adapted basis associated to the exact sequence (B, 2) in the case $K = K_1 + K_2$. Since $(F_1 + F_2)(\omega'_1) = F_1(\omega'_1)$,

$$\begin{aligned} \Phi_{K_1+K_2}(\xi \otimes F_1(\omega'_1)^*) &= (-1)^{\frac{(n_0+n_1+n_2)(n_0+n_1+n_2+1)}{2}} \\ &\quad \cdot (\xi \wedge (\eta'_1 \wedge \eta_2)) \otimes (P_{V_1}(\eta'_1) \wedge P_{V_2}(\eta_2) \wedge \omega'_1)^*. \end{aligned}$$

Since $\eta_2 \in \Lambda^{n_2}(\ker T_{K_1+K_2})$ and $\omega'_1 \in \Lambda^{n_0}V_1$,

$$P_{V_1}(\eta'_1) \wedge \omega'_1 \wedge P_{V_2}(\eta_2) = (-1)^{n_0 n_2} \cdot P_{V_1}(\eta'_1) \wedge P_{V_2}(\eta_2) \wedge \omega'_1. \quad (\text{B, 14})$$

Therefore, to prove that $\Phi_1 \circ \Phi_{K_1}(\xi \otimes F(\omega'_1)^*)$ and $\Phi_{K_1+K_2}(\xi \otimes F_1(\omega'_1)^*)$ are equal it remains to observe that

$$(-1)^{\frac{(n_0+n_1)(n_0+n_1+1)}{2} + n_1 n_2 + \frac{n_2(n_2+1)}{2}} = (-1)^{\frac{(n_0+n_1+n_2)(n_0+n_1+n_2+1)}{2} + n_0 n_2}. \quad \square$$

Remark. We have been so explicit in the last proof to show that the sign conventions are essential. If we had rather used the isomorphisms of (B, 6), we would have had a problem with signs. This is because the factor $(-1)^{n_0 n_2}$ in (B, 14) would not have cancelled out.

The determinant line bundle. The considerations in the last paragraph enable us to define the determinant line bundle over the space of Fredholm operators.

Theorem (B, 1.19). *There exists a canonical line bundle $\text{Det} \rightarrow \mathcal{F}$ with fibres $\text{Det}_T = \det T$ and the following property: For every finite dimensional Hilbert space V and every open subset $\mathcal{U} \subset \mathcal{F}$ such that there exists a stabilizer $K : \mathcal{U} \rightarrow \mathcal{L}(V, H_2)$, the fibrewise isomorphisms*

$$\Phi_{K,T} : \det T \longrightarrow \det(\ker T_K) \otimes (\det V)^*, \quad T \in \mathcal{U}, \quad (\text{B, 15})$$

given by Proposition (B, 1.17) yield a well-defined bundle isomorphism

$$\Phi_K : \text{Det}|_{\mathcal{U}} \longrightarrow \det(\text{Ker}_K) \otimes (\det V)^*|_{\mathcal{U}}.$$

Proof. Let $T_0 \in \mathcal{F}$, and consider a stabilizer K over some open neighbourhood \mathcal{U} of T_0 . Then we use the isomorphisms (B, 15) to pull back the line bundle structure of $\det(\text{Ker}_K) \otimes (\det V)^*|_{\mathcal{U}}$ to $\bigcup_{T \in \mathcal{U}} \det T$. Note that according to Lemma (B, 1.12), we can cover $\bigcup_{T \in \mathcal{F}} \det T$ in this way. Yet, we

still have to prove that the thus obtained line bundles patch together and do not depend on the stabilizers we have chosen. For this it suffices to check that given two stabilizers $K_1 : \mathcal{U} \rightarrow \mathcal{L}(V_1, H_2)$ and $K_2 : \mathcal{U} \rightarrow \mathcal{L}(V_2, H_2)$ over a common subset $\mathcal{U} \subset \mathcal{F}$, there exists a bundle isomorphism

$$\Phi_{12} : \det(Ker_{K_1}) \otimes (\det V_1)^*|_{\mathcal{U}} \rightarrow \det(Ker_{K_2}) \otimes (\det V_2)^*|_{\mathcal{U}}$$

such that for every $T \in \mathcal{U}$ the following diagram commutes:

$$\begin{array}{ccc} \det(ker T_{K_1}) \otimes (\det V_1)^* & \xrightarrow{\Phi_{12,T}} & \det(ker T_{K_2}) \otimes (\det V_2)^* \\ & \nwarrow \Phi_{K_1,T} \quad \nearrow \Phi_{K_2,T} & \\ & \det T & \end{array} .$$

It is immediately clear from (B, 9) that in the notation of Proposition (B, 1.18), letting $\Phi_{12} := \Phi_2^{-1} \circ \Phi_1$ makes the above diagram commutative. Moreover, the explicit formula (B, 13) shows that the collection of fibrewise maps yields, in fact, a *bundle* isomorphism Φ_{12} . This completes the proof. \square

Definition (B, 1.20). Let $T : X \rightarrow \mathcal{F}(H_1, H_2)$ be a continuous family of Fredholm operators, where X is an arbitrary topological space. Then the pullback bundle

$$\det T := T^*(Det) \longrightarrow X$$

is called the *determinant line bundle of the family T*. If $U \subset X$ is an open set such that there exists a stabilizer $K : T(U) \rightarrow \mathcal{L}(H_1 \oplus V, H_2)$ we shall call $K \circ T$ a *stabilizer of T over U* (and usually denote it also by K).

Remark. In particular, the line bundle $\det T \rightarrow X$ is defined in the following setting: Let $\{T_x\}_{x \in X}$ be a family of closed, densely defined Fredholm operators in a separable Hilbert space H such that $\text{dom } T_x = W$ is independent of x . Moreover, we assume that the induced graph norms on W are all equivalent. Then T can be regarded as a map $T : X \rightarrow \mathcal{F}(W, H)$, and thus the above definition applies provided that the map is continuous.

Now suppose that $T : X \rightarrow \mathcal{F}(H_1, H_2)$ is continuous where X is *compact*. According to Lemma (B, 1.12), we can cover X with finitely many U_i such that there exist finite dimensional subspaces $V_i \subset H_2$ such that $K_i := (V_i \hookrightarrow H_2)$ is a stabilizer over U_i . It is then clear that letting $V := \bigoplus_i V_i$ we may take $V \hookrightarrow H_2$ as a stabilizer over X . Hence,

Lemma (B, 1.21). *If X is compact and $T : X \rightarrow \mathcal{F}(H_1, H_2)$ is continuous, then there exists a finite dimensional subspace $V \subset H_2$ such that the constant $K := (V \hookrightarrow H_2)$ defines a stabilizer of T over X . In particular, the determinant line bundle $\det T \rightarrow X$ is globally isomorphic to*

$$\det(\ker T_K) \otimes (\det V)^* \longrightarrow X.$$

Appendix C

Spectral Flow and Orientation Transport

In this appendix, we summarize the definition and elementary properties of the spectral flow assigned to a path of self-adjoint elliptic operators. We shall not attempt to give the most general definition and restrict ourselves to the special cases which will actually occur in the applications we have in mind. However, this requires some nontrivial considerations about how the spectrum changes along a path of self-adjoint operators. We will only state the results we need referring to the literature for proofs.

In the second part we shall then see how to extract a number—the so-called orientation transport—from the determinant line bundle of a path of self-adjoint operators. We will then prove a formula relating this notion with the spectral flow. Due to the conventions we have used in Appendix B there are some subtleties about signs. Therefore, we include all proofs concerning the orientation transport.

1 Spectral flow

The spectral flow is assigned to a continuous path of bounded self-adjoint Fredholm operators by counting with multiplicity the number of eigenvalues passing through zero in the positive direction (see Fig. C.1). The original definition due to M. Atiyah and G. Lusztig is via the intersection number of the spectrum's graph with the zero line (cf. Atiyah et al. [3]). To get a rigorous definition one possibly has to perturb the path slightly in order to make certain that the intersection number is well-defined.

Remark. Nowadays there are more satisfactory, yet logically equivalent definitions of the spectral flow. J. Phillips' approach [49] is perhaps the

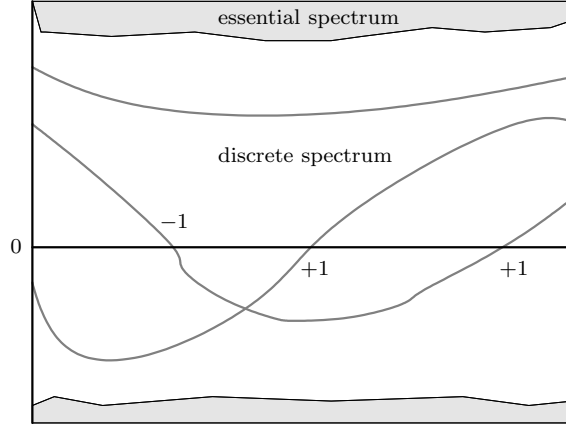


Figure C.1: Spectral flow

most appealing. An extensive treatment for paths of unbounded self-adjoint operators can be found in Booß-Bavnbek et al. [7]. Nevertheless, we will follow the original ideas of Atiyah since they are better suited for the actual computations which occur in this thesis.

Paths of Hermitian matrices. Before we can define the spectral flow, we have to recall some results from finite dimensional perturbation theory.

Let $\{A_t\}_{t \in [a,b]}$ be a path of self-adjoint operators in a finite dimensional \mathbb{K} -Hilbert space H . Modulo the choice of a basis of H , the path $\{A_t\}_{t \in [a,b]}$ consists of Hermitian¹ matrices. We want to find suitable paths of eigenvalues parametrizing the graph of the path's spectrum, i.e., the subset

$$\bigcup_{t \in [a,b]} \{t\} \times \text{spec}(A_t) \subset [a,b] \times \mathbb{R}$$

If A_t is a C^k -path such that there are only *simple* eigenvalues, then the implicit function theorem ensures that it is possible to find C^k -functions parametrizing the graph. In addition, it is then easy to see that there exist C^k -families of corresponding eigenvectors.

Whenever an eigenvalue has higher multiplicity, the situation becomes much more delicate. Nevertheless, there is one comparatively simple observation: We can find continuous paths of eigenvalues whenever A_t is continuous in t . Repeating each eigenvalue according to its multiplicity we only have to number the eigenvalues in ascending order and the paths obtained in this

¹In what follows, “Hermitian” has to be replaced with “symmetric” if $\mathbb{K} = \mathbb{R}$.

way turn out to be continuous (cf. Kato [23], Sec. II.5.2). If A_t is assumed to be C^k , smooth or analytic, it is natural to ask whether the parametrization of the spectrum's graph can be chosen to have a corresponding regularity. However, according to classical results due to Rellich, this is only partially true. We shall now recall some of those results.

Theorem (C, 1.1). (cf. [23], Thm. II.6.1). *Let A_t be a real analytic path of Hermitian matrices. Then there are two sets of real analytic families representing the repeated eigenvalues and the corresponding eigenvectors respectively.*

Theorem (C, 1.2). (cf. [23], Thm. II.6.8). *Assume that A_t is a C^1 -path of Hermitian matrices. Then there exist continuously differentiable functions λ_i representing the repeated eigenvalues.*

It should be noted that the proof of the second theorem is rather complicated and does not carry over to higher orders of differentiability. Moreover, we cannot hope that there exist corresponding C^1 -families of eigenvectors. This is illustrated by a famous example due to Rellich (cf. Kato [23], Ex. II.5.3). It is not difficult, though, to show that the total projection on the eigenspaces near a k -fold eigenvalue is continuously differentiable:

Theorem (C, 1.3). (cf. [23], Thm. II.5.4). *Let A_t be a C^1 -path of Hermitian matrices, and suppose that λ is a k -fold eigenvalue of A_{t_0} . If $\lambda_1, \dots, \lambda_k$ denote C^1 -functions representing eigenvalues of A_t such that $\lambda_i(t_0) = \lambda$, then the total projection*

$$\text{Proj}_{\ker(A-\lambda_1)} + \dots + \text{Proj}_{\ker(A-\lambda_k)}$$

is continuously differentiable near t_0 .

In addition to these well-known results, we want to point out a recent result due to Alekseevsky, Kriegel, Michor and Losik:

Theorem (C, 1.4). (cf. [2], Thm. 7.6). *Let A_t be a smooth family of Hermitian matrices such that no two of the continuous eigenvalues meet of infinite order at any t if they are not equal for all t . Then all the eigenvalues and all eigenvectors can be chosen smoothly in t on the whole parameter domain.*

Using the proof of the last results as a guideline, we now want to consider an example which we will need in the main part of this thesis.

Example (C, 1.5). Let $\{A_t\}_{t \in [-1,1]}$ be a C^4 -path of self-adjoint operators acting on a finite dimensional Hilbert space H and suppose that $A_0 = 0$ and that the eigenvalues of A'_0 are simple. Then the following holds:

- (i) The eigenvalues of A_t near 0 can be parametrized by C^2 -functions λ_i , and there exist corresponding C^2 -maps of normalized eigenvectors v_i .
- (ii) If in addition one of the eigenvalues of A'_0 vanishes then the corresponding C^2 -path of eigenvalues, say λ , satisfies

$$\lambda''(0) = \frac{d}{dt} \Big|_{t=0} \langle A'_t v_t, v_t \rangle, \quad (\text{C, 1})$$

where v_t is the path of eigenvectors associated to λ near 0.

Proof. Since $A_0 = 0$ and A_t is C^4 , we can use the Taylor approximation to write

$$A_t = A'_0 t + \frac{A_0^{(2)}}{2} t^2 + \frac{A_0^{(3)}}{6} t^3 + R(t),$$

with the Taylor remainder $R(t)$, which is C^4 and satisfies $\|R(t)\| \leq C|t|^4$ for all t . This implies that $\frac{R(t)}{t}$ is a C^2 -map with a zero of order 3 in 0. We can thus write $A_t = tB_t$, where B_t is a C^2 -path of self-adjoint operators on H satisfying $B_0 = A'_0$. From Theorem (C, 1.2) we thus obtain C^1 -functions μ_i parametrizing the eigenvalues of B_t . Due to our assumptions, each $\mu_i(0)$ is a simple eigenvalue of $B_0 = A'_0$ so that we can find $\varepsilon < 1$ such that all $\mu_i(t)$ are simple eigenvalues of B_t for each $|t| < \varepsilon$. Therefore, by virtue of the implicit function theorem, each $\mu_i(t)$ is necessarily a C^2 -function on $(-\varepsilon, \varepsilon)$. This shows in particular that $\ker(B_t - \mu_i(t))$ forms a C^2 -line bundle over $(-\varepsilon, \varepsilon)$. Choosing normalized sections $v_i(t)$ produces C^2 -paths of eigenvectors of B_t . Defining C^2 -functions $\lambda_i(t) := t\mu_i(t)$ we now observe that for $|t| < \varepsilon$,

$$A_t v_i(t) = t B_t v_i(t) = t \mu_i(t) v_i(t) = \lambda_i(t) v_i(t).$$

Therefore, each λ_i is a C^2 -path of eigenvalues of A_t near 0 with corresponding C^2 -path of eigenvectors v_i . This proves part (i).

Now assume that $\lambda = \lambda_{i_0}$ satisfies $\lambda'(0) = 0$. For any $t \in (-\varepsilon, \varepsilon)$, we then have $A_t v_t = \lambda(t) v_t$ with a C^2 -path of eigenvectors v_t of unit length. Differentiating this yields

$$A'_t v_t + A_t v'_t = \lambda'(t) v_t + \lambda(t) v'_t. \quad (\text{C, 2})$$

Taking the inner product with v_t results in

$$\lambda'(t) = \langle A'_t v_t, v_t \rangle + 2 \operatorname{Re} \langle A_t v_t, v'_t \rangle,$$

where we have used self-adjointness of A_t . Differentiating again and evaluating at $t = 0$, we infer that

$$\lambda''(0) = \frac{d}{dt}\bigg|_{t=0} \langle A'_t v_t, v_t \rangle + 2 \operatorname{Re} (\langle A'_0 v_0, v'_0 \rangle + \langle A_0 v'_0, v'_0 \rangle + \langle A_0 v_0, v''_0 \rangle).$$

Since $A_0 = 0$, $\lambda(0) = 0$ and $\lambda'(0) = 0$ we deduce from (C, 2) that $A'_0 v_0 = 0$. Therefore, the three summands on the right hand side of the above equation vanish, and this proves part (ii). \square

Paths of self-adjoint operators. After this short digression into finite dimensional Hilbert space theory, we return to the setting we actually have in mind. Let $\{T_t\}_{t \in [a,b]}$ be a family of unbounded, self-adjoint operators² in a separable \mathbb{K} -Hilbert space H . Assume that there exists a Hilbert space W such that all T_t have W as a common domain with graph norm equivalent to the given norm on W , i.e., we suppose that T_t is a function with values in the set

$$\mathcal{L}_{sa}(W, H) := \{T \in \mathcal{L}(W, H) \mid T \text{ self-adjoint operator in } H\}. \quad (\text{C, } 3)$$

In contrast to that, let

$$\mathcal{L}_{sym}(W, H) := \{T \in \mathcal{L}(W, H) \mid T \text{ symmetric operator in } H\}. \quad (\text{C, } 4)$$

Recall that $\mathcal{L}_{sym}(W, H)$ is a Banach space if endowed with the operator norm topology. Due to the Kato-Rellich Theorem (cf. [23], Thm. V.4.3), $\mathcal{L}_{sym}(W, H)$ contains $\mathcal{L}_{sa}(W, H)$ as an open subset. Therefore, in the case at hand, we may speak of continuity and differentiability of the family T_t with respect to the operator norm topology on $\mathcal{L}_{sa}(W, H)$.

Moreover, we assume that W embeds compactly in H which ensures that each T_t has discrete spectrum consisting of real eigenvalues with finite multiplicity.

The following result makes a precise definition of the spectral flow possible. It is a consequence of Theorem (C, 1.2) and Theorem (C, 1.3). A proof can be found in Robbin & Salamon [50], Cor. 4.29.

Theorem (C, 1.6) (Kato's Selection Theorem). *Suppose that T_t is continuously differentiable. Let $t_0 \in [a, b]$ and $c > 0$ such that $\pm c \notin \operatorname{spec}(T_{t_0})$, and let n be the dimension of the subspace spanned by eigenvectors corresponding to eigenvalues in $(-c, c)$. Then there exists a constant $\varepsilon > 0$ and C^1 -functions*

$$\lambda_i : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow (-c, c), \quad i \in \{1, \dots, n\}$$

²In our applications, T_t will be a family of formally self-adjoint, elliptic operators of constant order m , i.e., operators in $H = L^2$ with domain $W = L^2_m$ (cf. the discussion in App. A, Sec. 2).

with the following properties:

- $\lambda_i(t) \in \text{spec}(T_t)$
- $\lambda'_i(t) \in \text{spec}(P_i(t) \circ T'_t \circ P_i(t))$, where $P_i(t) = \text{Proj}_{\ker(T_t - \lambda_i(t))}$.
- If $\lambda \in \text{spec}(T_t) \cap (-c, c)$ with corresponding spectral projection P and $\mu \in \text{spec}(P \circ T'_t \circ P)$ is an eigenvalue of multiplicity m , then there are precisely m indices i_1, \dots, i_m such that $\lambda_{i_j}(t) = \lambda$ and $\lambda'_{i_j}(t) = \mu$ for $j = 1, \dots, m$.

Definition (C, 1.7). Assume that $T : [a, b] \rightarrow \mathcal{L}_{sa}(W, H)$ is continuously differentiable. Then we define the *crossing operator* of T at $t \in [a, b]$ by letting

$$C_T(t) := \text{Proj}_{\ker T_t} \circ T'_t|_{\ker T_t} : \ker T_t \rightarrow \ker T_t .$$

The path T is called *transversal* if $C_T(t)$ is invertible for each $t \in [a, b]$. Furthermore, T is said to have *simple crossings* if $\dim \ker T_t \leq 1$ for every $t \in [a, b]$.

Remark. Whenever T is a transversal path of self-adjoint operators, Kato's Selection Theorem ensures that the graph $\bigcup_{t \in [a, b]} \{t\} \times \text{spec}(T_t)$ intersects the line $[a, b] \times \{0\}$ transversally. This explains the chosen terminology.

An application of Sard's Theorem yields

Theorem (C, 1.8). (cf. [50], Thm. 4.22). Suppose $T : [a, b] \rightarrow \mathcal{L}_{sa}(W, H)$ is continuously differentiable. Then the path $T + \delta$ is transversal for almost every $\delta \in \mathbb{R}$.

Definition (C, 1.9). Let $T : [a, b] \rightarrow \mathcal{L}_{sa}(W, H)$ be a continuously differentiable path. Since T_a and T_b have discrete spectrum, there exists $\delta > 0$ such that the operators $T_a + \varepsilon$ and $T_b + \varepsilon$ are invertible for all $0 < \varepsilon \leq \delta$. According to the above result there exists ε among these such that the path $T + \varepsilon$ is transversal. We define the *spectral flow* of T by letting

$$\text{SF}(T) := \sum_{t \in (a, b)} \text{sign } C_{T+\varepsilon}(t) ,$$

where “sign” denotes the signature of a Hermitian endomorphism, i.e., the number of positive eigenvalues minus the number of negative ones.

By virtue of Kato's Selection Theorem, the above definition is independent of the choices made provided that δ is chosen sufficiently small.

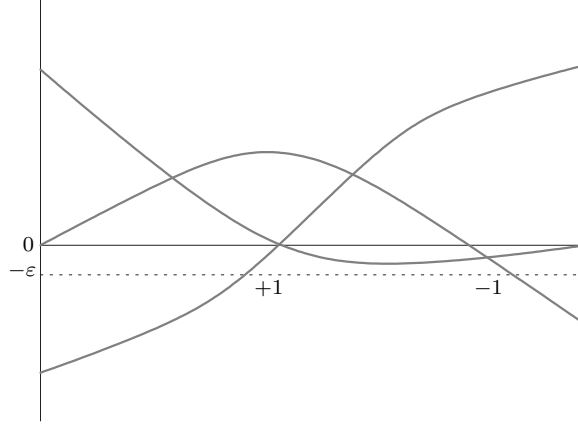


Figure C.2: Convention for counting zero eigenvalues at the endpoints

If ε can be chosen in such a way that in addition, $T + \varepsilon$ has only simple crossings, then the spectral flow of T is clearly given by

$$\text{SF}(T) = \sum_{t \in (a,b)} \text{sgn} \langle T'_t v_t, v_t \rangle ,$$

where each v_t is a vector spanning $\ker(T_t + \varepsilon)$. Here, “sgn” denotes the sign whereas “sign” is reserved for the signature of a Hermitian endomorphism.

Remark. Observe that our definition has a built-in convention of how to treat zero eigenvalues of T_a and T_b : Instead of counting crossings with the zero line, we take the intersection number of $\bigcup_{t \in [a,b]} \{t\} \times \text{spec}(T_t)$ with the line $[a, b] \times \{-\varepsilon\}$ (cf. Fig. C.2). Moreover, note that this corresponds to subtracting negative eigenvalues of $C_T(a)$ and adding positive eigenvalues of $C_T(b)$, i.e.,

$$\begin{aligned} \sum_{t \in (a,b)} \text{sign } C_{T+\varepsilon}(t) &= \sum_{t \in (a,b)} \text{sign } C_T(t) - \#\{\lambda \in \text{spec } C_T(a) \mid \lambda < 0\} \\ &\quad + \#\{\lambda \in \text{spec } C_T(b) \mid \lambda > 0\}. \end{aligned}$$

Properties of the spectral flow. For any pair $T_1 : [a, b] \rightarrow \mathcal{L}_{sa}(W_1, H_1)$ and $T_2 : [a, b] \rightarrow \mathcal{L}_{sa}(W_2, H_2)$ of paths of self-adjoint operators, we can form the direct sum

$$T_1 \oplus T_2 : [a, b] \rightarrow \mathcal{L}_{sa}(W_1 \oplus W_2, H_1 \oplus H_2) .$$

Given $T_1 : [a, b] \rightarrow \mathcal{L}_{sa}(W, H)$ and $T_2 : [b, c] \rightarrow \mathcal{L}_{sa}(W, H)$ satisfying $T_1(b) = T_2(b)$, we can also build the concatenation $T_1 \# T_2 : [a, c] \rightarrow \mathcal{L}_{sa}(W, H)$,

defined by

$$(T_1 \# T_2)(t) := \begin{cases} T_1(t) & \text{if } t \in [a, b] \\ T_2(t) & \text{if } t \in [b, c] . \end{cases}$$

Proposition (C, 1.10). (cf. [50], Thm. 4.23). *Let T_1 and T_2 be continuously differentiable paths of self-adjoint operators in H with domain W .*

- (i) *If T_1 is a constant family, then $\text{SF}(T_1) = 0$.*
- (ii) *If T_1 and T_2 are homotopic relative endpoints, then $\text{SF}(T_1) = \text{SF}(T_2)$.*
- (iii) *$\text{SF}(T_1 \oplus T_2) = \text{SF}(T_1) + \text{SF}(T_2)$, where we allow T_1 and T_2 to be defined in different Hilbert spaces.*
- (iv) *$\text{SF}(T_1 \# T_2) = \text{SF}(T_1) + \text{SF}(T_2)$, whenever the left-hand side is well-defined.*

Remark. It is not difficult to see that a homotopy invariant on the space of C^1 -paths gives also rise to a homotopy invariant for continuous paths. Thus we could go on and define the spectral flow in this more general setting like, for example, in Sec. 4 of [50]. However, the situation we shall actually encounter does not require any further considerations so that we refer to the literature for a more extensive treatment.

2 Orientation transport

Let H be a separable \mathbb{R} -Hilbert space and let $W \subset H$ be a dense subspace. We denote by

$$\mathcal{F}_{sa}(W, H) := \mathcal{L}_{sa}(W, H) \cap \mathcal{F}(W, H)$$

the space of closed, unbounded, self-adjoint Fredholm operators with domain W . Note that $\mathcal{F}_{sa}(W, H)$ is an open subset of $\mathcal{L}_{sym}(W, H)$ if we use the operator norm. We now consider the restriction of the determinant line bundle $\text{Det} \rightarrow \mathcal{F}(W, H)$ of Appendix B to $\mathcal{F}_{sa}(W, H)$. For every $T \in \mathcal{F}_{sa}(W, H)$ we have $(\text{im } T)^\perp = \ker T$. Using the Knudsen-Mumford sign conventions (B, 7), we thus have:

Lemma (C, 2.1). *For every $T \in \mathcal{F}_{sa}(W, H)$ let $n_0(T) := \dim(\ker T)$. Then there is a canonical isomorphism*

$$\Psi_T : \text{Det}_T = \det(\ker T) \otimes \det(\text{coker } T)^* \rightarrow \mathbb{R}, \quad (\text{C, 5})$$

given by

$$\Psi_T(\xi \otimes \omega^*) := (-1)^{\frac{n_0(T)(n_0(T)+1)}{2}} \omega^*[\xi].$$

Remark. Note that this lemma does not necessarily imply that the line bundle $Det|_{\mathcal{F}_{sa}} \rightarrow \mathcal{F}_{sa}$ is trivial since the collection $\{\Psi_T\}_{T \in \mathcal{F}_{sa}}$ need not give rise to a *continuous* map $Det|_{\mathcal{F}_{sa}} \rightarrow \mathcal{F}_{sa} \times \mathbb{R}$.

Orientation transport. Let $T : [a, b] \rightarrow \mathcal{F}_{sa}(W, H)$ be a continuous path. Since $[a, b]$ is contractible, there exists a trivialization of the determinant line bundle $\det T \rightarrow [a, b]$. This induces an isomorphism

$$\Psi_{T_a T_b} : \det T_a \rightarrow \det T_b. \quad (\text{C, 6})$$

By concatenation with the isomorphisms of (C, 5) at the endpoints T_a and T_b we get a chain of isomorphisms

$$\mathbb{R} \xrightarrow{\Psi_{T_a}^{-1}} \det T_a \xrightarrow{\Psi_{T_a T_b}} \det T_b \xrightarrow{\Psi_{T_b}} \mathbb{R}.$$

It is immediate that the parity of the isomorphism $\mathbb{R} \rightarrow \mathbb{R}$ given in this way is independent of the particular choice of trivialization of $\det T \rightarrow [a, b]$. Hence we may define:

Definition (C, 2.2). Let $T : [a, b] \rightarrow \mathcal{F}_{sa}(W, H)$ be a continuous path, and let $\Psi_{T_a T_b} : \det T_a \rightarrow \det T_b$ be an isomorphism induced by a trivialization of $\det T \rightarrow [a, b]$. Then

$$\varepsilon(T) := \text{sgn}(\Psi_{T_b} \circ \Psi_{T_a T_b} \circ \Psi_{T_a}^{-1}(1))$$

is called the *orientation transport* along T .

We are now going to derive an alternative formula for the orientation transport in the case of a continuous path $T : [a, b] \rightarrow \mathcal{F}_{sa}(W, H)$ which has invertible endpoints. Let $K : [a, b] \rightarrow \mathcal{L}(W \oplus V, H_2)$ be a stabilizer of T . Modulo the canonical isomorphism $\det T \cong \det(\ker T_K) \otimes (\det V)^*$, the isomorphisms (C, 5) at the endpoints T_a and T_b ,

$$\Psi_{K_t} : \det(\ker(T_K)_t) \otimes (\det V)^* \longrightarrow \mathbb{R}, \quad t \in \{a, b\},$$

are given by

$$\Psi_{K_t}(\eta \otimes P_V(\eta)^*) = (-1)^{\frac{n(n+1)}{2}}, \quad (\text{C, 7})$$

where $n := \dim V$. The orientation transport along T is given by

$$\varepsilon(T) = \text{sgn}(\Psi_{K_b} \circ \Psi_{T_a T_b} \circ \Psi_{K_a}^{-1}(1)),$$

with an isomorphism

$$\Psi_{T_a T_b} : \det(\ker(T_K)_a) \otimes (\det V)^* \rightarrow \det(\ker(T_K)_b) \otimes (\det V)^*$$

induced by a trivialization of $\det(\ker T_K) \otimes (\det V)^* \rightarrow [a, b]$. Clearly, choosing a trivialization of $\ker T_K \rightarrow [a, b]$ yields an isomorphism

$$\Psi_a^b : \ker(T_K)_a \rightarrow \ker(T_K)_b,$$

and we may then take

$$\Psi_{T_a T_b}(\eta \otimes \omega^*) := \Psi_a^b(\eta) \otimes \omega^*.$$

Formula (C, 7) implies that for any basis $\eta \in \det \ker(T_K)_a$,

$$\Psi_{K_b} \circ \Psi_{T_a T_b} \circ \Psi_{K_a}^{-1}(1) = (-1)^{\frac{n(n+1)}{2}} \cdot \Psi_{K_b}(\Psi_a^b(\eta) \otimes P_V(\eta)^*). \quad (\text{C, 8})$$

Now, to use (C, 7) for $t := b$, we observe that

$$\Psi_a^b(\eta) \otimes P_V(\eta)^* = \det(\Phi_V) \cdot \Psi_a^b(\eta) \otimes [P_V \circ \Psi_a^b(\eta)]^*, \quad (\text{C, 9})$$

where $\Phi_V : V \rightarrow V$ is defined by the following commutative diagram:

$$\begin{array}{ccc} \ker(T \oplus K)_a & \xrightarrow{\Psi_a^b} & \ker(T \oplus K)_b \\ \downarrow P_V & & \downarrow P_V \\ V & \xrightarrow{\Phi_V} & V \end{array} \quad (\text{C, 10})$$

Here, the projection $P_V : \ker(T_K)_t \rightarrow V$ is an isomorphism because T_t is invertible for $t \in \{a, b\}$. Inserting (C, 9) in (C, 8), we conclude

$$\Psi_{K_b} \circ \Psi_{T_a T_b} \circ \Psi_{K_a}^{-1}(1) = (-1)^{\frac{n(n+1)}{2}} \cdot (-1)^{\frac{n(n+1)}{2}} \cdot \det \Phi_V = \det \Phi_V.$$

Hence, we have proved:

Lemma (C, 2.3). *Let $T : [a, b] \rightarrow \mathcal{F}_{sa}(W, H)$ be a continuous path with invertible endpoints, and let $K : [a, b] \rightarrow \mathcal{L}(W \oplus V, H)$ be a stabilizer of T , and define $\Phi_V : V \rightarrow V$ via the commutative diagram (C, 10). Then*

$$\varepsilon(T) = \text{sgn}(\det \Phi_V).$$

Properties of the orientation transport. The orientation transport has some properties which are reminiscent of what we observed for the spectral flow in Proposition (C, 1.10). Using the definition of direct sum and concatenation as given there, we have:

Proposition (C, 2.4). *Let $T_0 \in \mathcal{F}_{sa}(W, H)$ be a self-adjoint Fredholm operator.*

- (i) *The orientation transport along the constant family $T_t := T_0$ is 1.*

- (ii) *If there exists $\delta > 0$ such that $T_t := T_0 + t$ is invertible for $0 < t \leq \delta$, then*

$$\varepsilon(T_t; 0 \leq t \leq \delta) = (-1)^{\dim(\ker T)}.$$

Moreover, if $T_0, T_1 : [a, b] \rightarrow \mathcal{F}_{sa}(W, H)$ are continuous paths.

- (iii) *If T_0 and T_1 are homotopic relative endpoints, then $\varepsilon(T_0) = \varepsilon(T_1)$.*
 (iv) *$\varepsilon(T_0 \oplus T_1) = \varepsilon(T_0) \cdot \varepsilon(T_1)$, where we allow T_0 and T_1 to be defined in different Hilbert spaces.*
 (v) *$\varepsilon(T_0 \# T_1) = \varepsilon(T_0) \cdot \varepsilon(T_1)$, whenever the left-hand side is well-defined.*

Proof. We only prove (ii) and (iii) since the other properties are straightforward. We start with the proof of (ii) using the same approach as in Lemma (C, 2.3). Letting $V := \ker T_0$, we consider

$$(T_V)_t : W \oplus V \rightarrow H, \quad (x, v) \mapsto T_t x + v, \quad t \in [0, \delta].$$

The determinant line bundle $\det T \rightarrow [0, \delta]$ obtains its line bundle structure via the natural isomorphism $\det T \cong \det(\ker T_V) \otimes (\det V)^*$. The bundle $\ker(T_V) \rightarrow [0, \delta]$ is canonically trivial via

$$[0, \delta] \times V \rightarrow \ker T_V, \quad (t, v) \mapsto (v, -tv).$$

Hence, we get an isomorphism

$$\Psi_0^\delta : \ker(T_V)_0 \rightarrow \ker(T_V)_\delta, \quad (v, 0) \mapsto (v, -\delta v)$$

which in turn induces an isomorphism on the level of determinants,

$$\begin{aligned} \Psi_{T_0 T_\delta} : \det(\ker(T_V)_0) \otimes (\det V)^* &\rightarrow \det(\ker(T_V)_\delta) \otimes (\det V)^*, \\ \xi \otimes \omega^* &\mapsto \Psi_0^\delta(\xi) \otimes \omega^*. \end{aligned}$$

Modulo the identification $\det T \cong \det(\ker T_V) \otimes (\det V)^*$, the isomorphisms (C, 5) at the endpoints T_0 and T_δ take the form

$$\begin{aligned} \Psi_{T_0} : \det(\ker(T_V)_0) \otimes (\det V)^* &\rightarrow \mathbb{R}, \quad \Psi_{T_0}(\xi \otimes \omega^*) = (-1)^{n + \frac{n(n+1)}{2}} \cdot \omega^*[\xi], \\ \Psi_{T_\delta} : \det(\ker(T_V)_\delta) \otimes (\det V)^* &\rightarrow \mathbb{R}, \quad \Psi_{T_\delta}(\eta \otimes P_V(\eta)^*) = (-1)^{\frac{n(n+1)}{2}}, \end{aligned}$$

where $n := \dim V$. Now, the orientation transport along T is given by

$$\varepsilon(T) = \operatorname{sgn}(\Psi_{T_\delta} \circ \Psi_{T_0 T_\delta} \circ \Psi_{T_0}^{-1}(1)).$$

From the explicit formula for Ψ_0^δ one readily gets that $P_V(\Psi_0^\delta(\xi)) = (-\delta)^n \cdot \xi$. This implies

$$\xi^* = (-\delta)^n \cdot P_V(\Psi_0^\delta(\xi))^*.$$

Hence, the expressions for Ψ_{T_0} and Ψ_{T_δ} imply that

$$\begin{aligned} \Psi_{T_\delta} \circ \Psi_{T_0 T_\delta} \circ \Psi_{T_0}^{-1}(1) &= (-1)^{n + \frac{n(n+1)}{2}} \cdot \Psi_{T_\delta}(\Psi_0^\delta(\xi) \otimes \xi^*) \\ &= (-1)^{n + \frac{n(n+1)}{2}} \cdot \Psi_{T_\delta}((- \delta)^n \cdot \Psi_0^\delta(\xi) \otimes P_V(\Psi_0^\delta(\xi))^*) \\ &= (-1)^{n + \frac{n(n+1)}{2}} \cdot (-\delta)^n \cdot (-1)^{\frac{n(n+1)}{2}} = \delta^n. \end{aligned}$$

Since $\delta > 0$, the orientation transport along T is 1.

To prove (iii), suppose that

$$S : [a, b] \times [0, 1] \rightarrow \mathcal{F}_{sa}(W, H)$$

is a homotopy connecting T_0 and T_1 and leaving the endpoints fixed. We then consider the determinant bundle $\det S \rightarrow [a, b] \times [0, 1]$. Note that then, we have the following isomorphism of line bundles:

$$\det T_0 = \det S|_{[a, b] \times \{0\}} \quad \text{and} \quad \det T_1 = \det S|_{[a, b] \times \{1\}}.$$

Since S leaves the endpoints fixed, we may define for arbitrary $s \in [0, 1]$

$$L_a := \det S(a, s) = \det(T_0)_a = \det(T_1)_a$$

and similarly L_b . As $[a, b] \times [0, 1]$ is contractible, there exists a nowhere vanishing section $\Psi : [a, b] \times [0, 1] \rightarrow \det S$. Restricting Ψ to $[a, b] \times \{0\}$ and $[a, b] \times \{1\}$, we get isomorphisms

$$\Psi^0 : L_a \rightarrow L_b \quad \text{and} \quad \Psi^1 : L_a \rightarrow L_b.$$

As both are induced by Ψ , they are homotopic in the space of isomorphisms $L_a \rightarrow L_b$. This shows that using Ψ^0 and Ψ^1 as in the definition to compute the orientation transport along T_0 and T_1 respectively, we get $\varepsilon(T_0) = \varepsilon(T_1)$. \square

Orientation transport and spectral flow. The above proposition suggests an important connection between the spectral flow and the orientation transport of a path of self-adjoint Fredholm operators. Since we defined the spectral flow only in a more restrictive context, we now consider C^1 -paths and require that W embeds compactly³ in H .

³Consequently, elements of $\mathcal{L}_{sa}(W, H)$ have compact resolvent, so that they are automatically Fredholm.

Theorem (C, 2.5). *Let $T : [a, b] \rightarrow \mathcal{L}_{sa}(W, H)$ be a continuously differentiable path. Then*

$$\varepsilon(T) = (-1)^{\text{SF}(T)}.$$

Proof. Recall that $\text{SF}(T)$ is defined as $\sum_{t \in (a, b)} \text{sign } C_{T+\delta}(t)$, where $C_{T+\delta}(t)$ is the crossing operator and $\delta > 0$ is such that $T + \delta$ is transversal and $T + t$ has invertible endpoints for $0 < t \leq \delta$. Employing (ii), (iii) and (v) of Proposition (C, 2.4), one straightforwardly shows that $\varepsilon(T) = \varepsilon(T + \delta)$ so that without loss of generality, we may assume that T is already a transversal family with invertible endpoints and drop δ from the notation. We are then to show that

$$\begin{aligned} \varepsilon(T) &= (-1)^{\sum_{t \in (a, b)} \text{sign } C_T(t)} \\ &= \prod_{t \in (a, b)} (-1)^{\text{sign } C_T(t)} = \prod_{t \in (a, b)} (-1)^{\dim(\ker T_t)}. \end{aligned} \quad (\text{C, 11})$$

Invoking (v) of Proposition (C, 2.4), we can clearly restrict to the case of a continuously differentiable path $T : [-1, 1] \rightarrow \mathcal{L}_{sa}(W, H)$ with the property that T_t is invertible for each $t \neq 0$. Under this assumption, $V := \ker T_0$ is a stabilizer of T over $[-1, 1]$. Since $\ker T_V \rightarrow [-1, 1]$ is a C^1 -vector bundle over a contractible space, we may choose trivializing C^1 -sections

$$e_i(t) : [-1, 1] \rightarrow \ker T_V.$$

Moreover, we can clearly achieve that $(e_1(0), \dots, e_n(0))$ is an orthonormal basis of V . Write

$$e_i(t) = (e_i(0) + w_i(t), v_i(t))$$

with appropriate C^1 -maps $w_i : [-1, 1] \rightarrow W$ and $v_i : [-1, 1] \rightarrow V$. Note that $w_i(t) \rightarrow 0$ and $v_i(t) \rightarrow 0$ as $t \rightarrow 0$.

We shall now compute $\varepsilon(T)$ via $\varepsilon(T|_{[-t_0, t_0]})$ by taking the limit $t_0 \rightarrow 0$. For this note that the orientation transport along $T|_{[-t_0, t_0]}$ is independent of t_0 because $T|_{[-1, -t_0]}$ and $T|_{[t_0, 1]}$ are paths of invertible operators, thus giving no contribution. We use Lemma (C, 2.3) to compute $\varepsilon(T|_{[-t_0, t_0]})$. One readily checks that the isomorphism $\Phi_V : V \rightarrow V$ given by the diagram (C, 10) in the situation at hand is uniquely determined by

$$\Phi_V(v_i(-t_0)) = v_i(t_0).$$

Hence, the orientation transport along $T|_{[-t_0, t_0]}$ is given by

$$\varepsilon(T|_{[-t_0, t_0]}) = \text{sgn det} \left(\langle v_i(-t_0), v_j(t_0) \rangle \right)_{ij}.$$

As we want to determine this sign by letting $t_0 \rightarrow 0$, we write—using that T is continuously differentiable—

$$T_t = T_0 + tT'_0 + o(t).$$

Since $(T_V)_t e_i(t) = 0$, $T_0 e_i(0) = 0$, and $tw_i(t) = o(t)$, we deduce that

$$0 = (T_V)_t (e_i(0) + w_i(t), v_i(t)) = T_0 w_i(t) + tT'_0 e_i(0) + v_i(t) + o(t).$$

Applying $P := \text{Proj}_V = \text{Proj}_{\ker T_0}$ to the above equation deletes the first term, and thus,

$$v_i(t) = -tPT'_0 e_i(0) + o(t)$$

From this we conclude that for small $t_0 > 0$,

$$\begin{aligned} \det \left(\langle v_i(-t_0), v_j(t_0) \rangle \right)_{ij} &= \det \left(\left\langle \frac{v_i(-t_0)}{t_0}, \frac{v_j(t_0)}{t_0} \right\rangle \right)_{ij} \\ &= \det \left(\left\langle PT'_0 e_i(0) + \frac{o(t_0)}{t_0}, -PT'_0 e_j(0) + \frac{o(t_0)}{t_0} \right\rangle \right)_{ij}. \end{aligned}$$

This expression allows to perform the limit $t_0 \rightarrow 0$, which produces

$$\begin{aligned} \varepsilon(T) &= \text{sgn} \det \left(\left\langle -PT'_0 e_i(0), PT'_0 e_j(0) \right\rangle \right)_{ij} \\ &= \text{sgn} \left((-1)^n \det(PT'_0 P)^2 \right) = (-1)^n. \end{aligned}$$

As $n = \dim(\ker T_0)$, equation (C, 11) is established. \square

Appendix D

Spin^c Manifolds

In this appendix we give a summary of the constructions related to so-called spin^c manifolds. First of all, we make some algebraic remarks concerning the group Spin^c and its representation theory. As we assume some familiarity with the definition of Clifford algebras, the Spin group, and their representation theory, the presentation in Section 1 will be rather sketchy, not containing proofs. We refer to the wide range of literature, in particular Lawson & Michelsohn [29], Ch. I, or Berline et al. [5], Ch. 3, for a more detailed treatment.

However, differences between spin^c and spin become more intriguing when we consider the geometric framework in Section 2. Here, we shall go in more detail since an understanding of the special nature of spin^c structures is an important prerequisite for studying Seiberg-Witten theory. Once we have established a suitable setting in which to define the so-called spin^c Dirac operator, the discussion of the related analytic properties proceeds in almost the same manner as for the spin Dirac operator. Hence, in Section 3, we will again simply state the results, giving references for the proofs.

Section 4 contains some material related to the question of how the spin^c Dirac operator depends on the metric. This will be needed when we compare the structures of Seiberg-Witten moduli spaces for different Riemannian metrics. It is placed here in order not to interrupt the line of argument in the main part of this thesis.

1 The group Spin^c

The Clifford algebra. Let (V, g) be a Euclidean vector space with corresponding Clifford algebra $\text{Cl}(V)$. If $V = \mathbb{R}^n$, we shall simply write Cl_n . Recall that $\text{Cl}(V)$ is the associative real algebra generated by 1 and elements

of V subject to the relations

$$v \cdot w + w \cdot v = -2g(v, w)1, \quad v, w \in V.$$

We denote the complexified Clifford algebra by

$$\text{Cl}^c(V) := \text{Cl}(V) \otimes \mathbb{C}.$$

Then $\text{Cl}^c(V)$ can be interpreted as the Clifford algebra of the complex vector space $V \otimes \mathbb{C}$ endowed with the complex bilinear extension of g .

The representation theory of the complex Clifford algebra turns out to be rather simple. Let $n = \dim V$.

- If $n = 2k$, then there exists up to isomorphism exactly one irreducible $\text{Cl}^c(V)$ -module, denoted by Δ , which has dimension 2^k .
- If $n = 2k + 1$, then there are two irreducible $\text{Cl}^c(V)$ -modules, Δ^+ and Δ^- , of dimension 2^k .

We recall how to distinguish the inequivalent representations in the case that n is odd. Let

$$\omega^c := i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n \tag{D, 1}$$

be the *complex volume element* of the Clifford algebra. Here, for a real number a , $\lfloor a \rfloor$ denotes the greatest integer smaller than a . Then ω^c is an involution (irrespective of the parity of n), i.e., $(\omega^c)^2 = 1$. Therefore, each $\text{Cl}^c(V)$ -module splits into ± 1 eigenspaces of ω^c . If n is odd, then ω^c is central and therefore, its eigenspaces are $\text{Cl}^c(V)$ -invariant. Then Δ^\pm is the irreducible module on which ω^c acts as $\pm \text{id}$.

Spin and Spin^c . Recall that the Spin group associated to (V, g) is defined by

$$\text{Spin}(V) := \{v_1 \cdots v_m \mid m \text{ even, } |v_i| = 1\} \subset \text{Cl}(V)^*,$$

where $\text{Cl}(V)^*$ denotes the group of units in $\text{Cl}(V)$. It turns out that $\text{Spin}(V)$ is a compact, connected Lie group which is simply connected if $\dim V \geq 3$. We shall always write $\text{Spin}_n := \text{Spin}(\mathbb{R}^n)$. The subspace V of $\text{Cl}(V)$ is invariant with respect to conjugation by elements of $\text{Spin}(V)$. Moreover, it turns out that for each $g \in \text{Spin}(V)$, the endomorphism

$$V \rightarrow V, \quad v \mapsto gvg^{-1}$$

is, in fact, in $\text{SO}(V)$ and that the Lie group homomorphism obtained in this way, say,

$$\xi_0 : \text{Spin}(V) \longrightarrow \text{SO}(V), \tag{D, 2}$$

is surjective with $\ker \xi_0 = \{\pm 1\}$. Therefore, ξ_0 gives a twofold covering of $\text{SO}(V)$ which is universal if $\dim V \geq 3$.

Definition (D, 1.1). The *complex spin group*, $\text{Spin}^c(V)$, is the group generated by $\text{Spin}(V)$ and U_1 inside the group $\text{Cl}^c(V)^*$. If $V = \mathbb{R}^n$, we write $\text{Spin}_n^c := \text{Spin}^c(\mathbb{R}^n)$.

Since U_1 lies in the center of $\text{Cl}^c(V)$ and $\text{Spin}(V) \cap U_1 = \{\pm 1\}$, it is clear that

$$\text{Spin}^c(V) = \text{Spin}(V) \times_{\mathbb{Z}_2} U_1,$$

where $\mathbb{Z}_2 = \{\pm 1\}$ acts diagonally. This action is free, hence $\text{Spin}^c(V)$ inherits the structure of a compact, connected real Lie group. The covering map $\xi_0 : \text{Spin}(V) \rightarrow \text{SO}(V)$ gives rise to an exact sequence of Lie groups

$$1 \longrightarrow U_1 \longrightarrow \text{Spin}^c(V) \xrightarrow{\xi_0^c} \text{SO}(V) \longrightarrow 1, \quad (\text{D}, 3)$$

where $\xi_0^c([g, z]) := \xi_0(g)$. Note that this is well-defined since $\ker \xi_0 = \mathbb{Z}_2$. Defining $\zeta^c([g, z]) := z^2$, we obtain another exact sequence

$$1 \longrightarrow \text{Spin}(V) \longrightarrow \text{Spin}^c(V) \xrightarrow{\zeta^c} U_1 \longrightarrow 1. \quad (\text{D}, 4)$$

The maps ξ_0^c and ζ^c induce a two sheeted covering of Lie groups

$$\xi : \text{Spin}^c(V) \rightarrow \text{SO}(V) \times U_1, \quad [g, z] \mapsto (\xi_0(g), z^2). \quad (\text{D}, 5)$$

Spin representation. Let us now turn to the representation theory of the group Spin^c .

Definition (D, 1.2). Let W be an irreducible $\text{Cl}^c(V)$ -module. By restricting the action of $\text{Cl}^c(V)$ we get a representation of $\text{Spin}^c(V)$ on W . The representation obtained in this way is called a *spin representation*.

Using the classification of irreducible Cl^c -modules and analysing the restriction to Spin^c yields the following:

- If n is even, then the decomposition $\Delta = \Delta^+ \oplus \Delta^-$ of the unique irreducible Cl^c -module is Spin^c -invariant, inducing the irreducible *half spin representations* of Spin^c ;
- if n is odd, then the two non-isomorphic, irreducible Cl^c -modules give rise to equivalent irreducible representations of Spin^c .

2 Spin^c structures

Let (M, g) be an oriented, n -dimensional Riemannian manifold, and let $P_{\text{SO}}(g)$ denote its principal SO_n -bundle of oriented, orthonormal frames. Before we define the notion of a spin^c structure, we first recall the definition of a spin manifold.

Definition (D, 2.1). A *spin structure* ε on M is a principal Spin_n -bundle $P_{\text{Spin}}(\varepsilon)$ together with a bundle map $\xi : P_{\text{Spin}}(\varepsilon) \rightarrow P_{\text{SO}}$, which is Spin -equivariant with respect to the two sheeted covering $\xi_0 : \text{Spin}_n \rightarrow \text{SO}_n$. Here, equivariance means that $\xi(pg) = \xi(p)\xi_0(g)$ for every $p \in P_{\text{Spin}}(\varepsilon)$ and every $g \in \text{Spin}_n$. The pair (M, ε) is called a *spin manifold*.

Imitating the above definition, we now introduce the notion of a spin^c structure:

Definition (D, 2.2). A *spin^c structure* on M , denoted by σ , consists of a principal Spin_n^c -bundle $P_{\text{Spin}^c}(\sigma)$ together with a bundle map $\xi^c : P_{\text{Spin}^c}(\sigma) \rightarrow P_{\text{SO}}$ which is Spin_n^c -equivariant with respect to the homomorphism $\xi_0^c : \text{Spin}_n^c \rightarrow \text{SO}_n$. The pair (M, σ) is called a *spin^c manifold*.

Another bundle is encoded in the definition of a spin^c manifold (M, σ) . Recall that to any principal bundle we can associate new principal bundles and vector bundles via group homomorphisms and representations of the structure group. Hence, via the map $\zeta^c : \text{Spin}_n^c \rightarrow \text{U}_1$ of (D, 4) we obtain the principal U_1 -bundle

$$P_{\text{U}_1}(\sigma) := P_{\text{Spin}^c}(\sigma) \times_{\zeta^c} \text{U}_1$$

which is the quotient of $P_{\text{Spin}^c}(\sigma) \times \text{U}_1$ with respect to $(p, z) \sim (pg, \zeta^c(g^{-1})z)$ endowed with the right action of U_1 on the second factor. Equivalently, we may consider the Hermitian line bundle

$$L(\sigma) := P_{\text{Spin}^c}(\sigma) \times_{\zeta^c} \mathbb{C}.$$

Definition (D, 2.3). The line bundle $L(\sigma)$ is called the *canonical line bundle* of the spin^c structure σ . Its topological first Chern class is called the *canonical class* and will be denoted by $c(\sigma) \in H^2(M; \mathbb{Z})$.

Remark.

- (i) In the main part of this thesis, we shall also refer to the image of $c(\sigma)$ in $H^2(M; \mathbb{R})$ as the canonical class of σ . As the meaning should be understood from the context, we can avoid a notational distinction like “ $c^{\text{top}}(\sigma)$ ” and “ $c^{\text{geom}}(\sigma)$ ”.

(ii) We note that a spin^c structure could equally well be defined by the following data:

- A principal U_1 -bundle $P_{U_1}(\sigma) \rightarrow M$,
- a principal Spin_n^c -bundle $P_{\text{Spin}^c}(\sigma) \rightarrow M$,
- a Spin_n^c -equivariant bundle map $\xi : P_{\text{Spin}^c}(\sigma) \rightarrow P_{\text{SO}} \times P_{U_1}(\sigma)$.

Here, $P_{\text{SO}} \times P_{U_1}(\sigma)$ denotes the fibre product of the bundles $P_{\text{SO}} \rightarrow M$ and $P_{U_1}(\sigma) \rightarrow M$. Note that $\xi : P_{\text{Spin}^c}(\sigma) \rightarrow P_{\text{SO}} \times P_{U_1}(\sigma)$ is a twofold covering.

Definition (D, 2.4). Two spin^c structures σ and σ' on M are called *equivalent* if there exists a bundle isomorphism $\Phi : P_{\text{Spin}^c}(\sigma) \rightarrow P_{\text{Spin}^c}(\sigma')$ inducing a commutative diagram:

$$\begin{array}{ccc}
 P_{\text{Spin}^c}(\sigma) & \xrightarrow{\Phi} & P_{\text{Spin}^c}(\sigma') \\
 \searrow \xi^c & & \swarrow \xi^{c'} \\
 & P_{\text{SO}} & \\
 \searrow & \downarrow & \swarrow \\
 & M &
 \end{array} \tag{D, 6}$$

The set of equivalence classes of spin^c structures shall be denoted by $\text{spin}^c(M)$.

Before we study possible topological obstructions to the existence of spin^c structures, let us first consider some examples.

Example (D, 2.5). *Each spin structure induces a canonical spin^c structure.* Given a spin manifold (M, ε) , we can form a Spin_n^c -bundle by letting

$$P_{\text{Spin}^c} := P_{\text{Spin}}(\varepsilon) \times_{\mathbb{Z}_2} U_1 ,$$

where U_1 denotes the trivial bundle $M \times U_1$, and \mathbb{Z}_2 acts diagonally. The map $\zeta = z^2 : U_1 \rightarrow U_1$ and the bundle map $P_{\text{Spin}}(\varepsilon) \rightarrow P_{\text{SO}}$ are \mathbb{Z}_2 -invariant thus giving a Spin_n^c -equivariant bundle map

$$\xi : P_{\text{Spin}}(\varepsilon) \times_{\mathbb{Z}_2} U_1 \longrightarrow P_{\text{SO}} \times U_1 .$$

Therefore, we obtain the so-called *canonical spin^c structure*, $\sigma(\varepsilon)$, on M . Clearly, the canonical line bundle $L(\sigma(\varepsilon))$ is the trivial bundle so that in particular, $c(\sigma(\varepsilon)) = 0$.

We state another example which is important in four dimensional Seiberg-Witten theory, especially when dealing with Hermitian or Kähler manifolds:

Example (D, 2.6). *Every almost complex manifold has a canonical spin^c structure.* Let M be a $2k$ -dimensional Riemannian manifold which admits a compatible almost complex structure, i.e., an orthogonal bundle map $J : TM \rightarrow TM$ with $J^2 = -\text{id}$. Then (TM, J) carries the structure of a complex vector bundle of rank k over M with an induced Hermitian metric. Therefore, the structure group of TM can be reduced to U_k , i.e., we can construct a principal U_k -bundle P_{U_k} on M such that (TM, J) is the vector bundle associated to P_{U_k} via the standard representation of U_k on \mathbb{C}^k . The inclusion $i : U_k \hookrightarrow \text{SO}_{2k}$ gives rise to a group homomorphism $(i, \det) : U_k \rightarrow \text{SO}_{2k} \times U_1$. One checks that the fundamental groups are related via $(i, \det)_* \pi_1(U_k) \subset \xi_* \pi_1(\text{Spin}_{2k}^c)$ so that there exists a unique lifting

$$\begin{array}{ccc} & & \text{Spin}_{2k}^c \\ & \nearrow j & \downarrow \xi \\ U_k & \xrightarrow{(i, \det)} & \text{SO}_{2k} \times U_1 \end{array}$$

We now obtain a canonical spin^c structure σ_J by letting

$$P_{\text{Spin}^c}(\sigma_J) := P_{U_k} \times_j \text{Spin}_{2k}^c.$$

It turns out that the canonical line bundle of the spin^c manifold (M, σ_J) is precisely the dual of K , the canonical line bundle of the almost complex manifold (M, J) , i.e.,

$$L(\sigma_J) = \Lambda^{k,0} TM = K^*.$$

Principal bundles and Čech cohomology. To understand the topological obstructions to the existence of spin^c structures, we briefly recall the interaction between the local description of principal bundles and Čech cohomology. For a more detailed exposition, the reader is referred to Hirzebruch [19], Ch. I.

Let $\{U_\alpha\}$ be a good open cover of a manifold M , i.e., a covering by open sets such that all intersections are contractible. Suppose $P \rightarrow M$ is a principal G -bundle, where G is a Lie group. Since P can be trivialized over $\{U_\alpha\}$, we obtain a corresponding family of transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G\}$ fulfilling the cocycle condition

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma. \quad (\text{D, } 7)$$

On the other hand, there is a well-known procedure to define a principal G -bundle given such a family of transition functions. Two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define isomorphic G -bundles if and only if there exists a family $\{\Phi_\alpha : U_\alpha \rightarrow G\}$ such that

$$g'_{\alpha\beta} = \Phi_\alpha g_{\alpha\beta} \Phi_\beta^{-1} \quad \text{on } U_\alpha \cap U_\beta. \quad (\text{D}, 8)$$

More generally, let $\mu : G \rightarrow H$ be a Lie group homomorphism and $Q \rightarrow M$ a principal H -bundle with transition functions $\{h_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow H\}$. A bundle map $\Phi : G \rightarrow H$ which is equivariant with respect to μ turns out to be the same as a family $\{\Phi_\alpha : U_\alpha \rightarrow H\}$ satisfying

$$h_{\alpha\beta} = \Phi_\alpha \mu(g_{\alpha\beta}) \Phi_\beta^{-1} \quad \text{on } U_\alpha \cap U_\beta. \quad (\text{D}, 9)$$

Let \underline{G} be the sheaf of germs of differentiable G -valued functions on M . Then formula (D, 7) is exactly the condition for the Čech 1-chain $\{g_{\alpha\beta}\}$ to define an element

$$[g_{\alpha\beta}] \in H^1(M; \underline{G}).^1$$

Furthermore, equation (D, 8) is equivalent to $[g_{\alpha\beta}] = [g'_{\alpha\beta}] \in H^1(M; \underline{G})$. Hence, there is a natural correspondence between the isomorphism classes of principal G -bundles and the first cohomology $H^1(M; \underline{G})$.

Example (D, 2.7). Consider the exponential sequence

$$0 \rightarrow \mathbb{Z} \longrightarrow \underline{\mathbb{R}} \xrightarrow{\exp(2\pi i x)} \underline{U}_1 \rightarrow 0.$$

As an exact sequence of sheaves it yields a long exact sequence in cohomology. The sheaf of germs of differentiable, \mathbb{R} -valued functions admits partitions of unity and therefore, $H^k(M; \underline{\mathbb{R}}) = 0$ for every $k \geq 1$.² This shows that for $k \geq 1$, the connecting homomorphism

$$\delta^k : H^k(M; \underline{U}_1) \rightarrow H^{k+1}(M; \mathbb{Z})$$

is an isomorphism. In particular, the set of isomorphism classes of principal U_1 -bundles is isomorphic to $H^2(M; \mathbb{Z})$. It turns out (cf. Wells [59], Sec. III.4)

¹If G is non-abelian, one can define $H^1(M; \underline{G})$ in the same way as for abelian G —with the difference that it will not be a group but only a pointed set (with the trivial G -bundle as a base point). The long exact cohomology sequence associated to an exact sequence of sheaves (see [19], Sec. I.2) then terminates at the H^2 level if a non-abelian group is involved. Note that this sequence is then an exact sequence of pointed sets.

²Note the difference between $H^k(M; \underline{\mathbb{R}})$ and the cohomology groups $H^k(M; \mathbb{R})$, associated to the sheaf \mathbb{R} of locally constant functions. The latter cohomology groups are isomorphic to the deRham cohomology. Hence, in general, $H^k(M; \mathbb{R}) \neq 0$.

that, if $P \rightarrow M$ is a U_1 -bundle with transition functions $\{\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow U_1\}$, then

$$\delta^1([\lambda_{\alpha\beta}]) = c_1(P),$$

the latter denoting first Chern class of P .

Local description of spin and spin^c structures. In this paragraph we shall treat only spin structures since the discussion for spin^c structures is completely analogous.

Let $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{SO}_n\}$ be a cocycle defining the SO_n bundle of M . A spin structure consists of a principal Spin_n -bundle $P_{\text{Spin}} \rightarrow M$, given by a cocycle $\{h_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Spin}_n\}$ and a bundle map $\xi : P_{\text{Spin}} \rightarrow P_{\text{SO}}$ which is equivariant with respect to the Lie group homomorphism $\xi_0 : \text{Spin}_n \rightarrow \text{SO}_n$. According to (D, 9), the map ξ is equivalently given by a family $\{\xi_\alpha : U_\alpha \rightarrow \text{SO}_n\}$ satisfying

$$g_{\alpha\beta}(x) = \xi_\alpha(x)\xi_0(h_{\alpha\beta}(x))\xi_\beta^{-1}(x), \quad x \in U_\alpha \cap U_\beta. \quad (\text{D}, 10)$$

Since all U_α are contractible and ξ_0 is a covering³ map, we can find maps $\Phi_\alpha : U_\alpha \rightarrow \text{Spin}_n$ such that

$$\xi_0 \circ \Phi_\alpha = \xi_\alpha.$$

Letting $h'_{\alpha\beta} := \Phi_\alpha h_{\alpha\beta} \Phi_\beta^{-1}$ we obtain a family of transition function defining a Spin_n -bundle $P'_{\text{Spin}} \rightarrow M$ and a bundle isomorphism $\Phi : P_{\text{Spin}} \rightarrow P'_{\text{Spin}}$. Hence, P'_{Spin} and $\xi' := \xi \circ \Phi^{-1}$ define a spin structure which is equivalent to the original one. However, equation (D, 10) is simplified since

$$g_{\alpha\beta} = \xi_\alpha \xi_0(h_{\alpha\beta}) \xi_\beta^{-1} = \xi_0(\Phi_\alpha) \xi_0(h_{\alpha\beta}) \xi_0(\Phi_\beta^{-1}) = \xi_0(h'_{\alpha\beta}).$$

Hence, modulo equivalence, a spin structure is always given by a cocycle $\{h_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Spin}_n\}$ lifting the family $\{g_{\alpha\beta}\}$ via ξ_0 . Note that this implies that the bundle map $\xi : P_{\text{Spin}} \rightarrow P_{\text{SO}}$ is given by the family $\{\xi_\alpha = \text{id} : U_\alpha \rightarrow \text{Spin}_n\}$. Mutatis mutandis, the same holds for spin^c structures. For brevity, transition functions with the above property will be called *fitting* cocycles.

Remark. Notice, however, that isomorphic Spin_n -bundles covering P_{SO} can give rise to nonequivalent spin structures if the corresponding diagram (D, 6) is not commutative. An example for this phenomenon is given by Milnor in [39] (see also Lawson & Michelsohn [29], II.1.14).

³When discussing spin^c structures, the corresponding map $\xi_0^c : \text{Spin}_n^c \rightarrow \text{SO}_n$ is a U_1 -fibration and thus also has the lifting property.

The set $\text{spin}^c(M)$. The above local description yields a possibility to analyse the set $\text{spin}^c(M)$ of all possible spin^c structures on a manifold M .

Proposition (D, 2.8). *Let M be a manifold admitting a spin^c structure. Then there exists a natural action*

$$\text{spin}^c(M) \times H^1(M; \underline{\mathbb{U}}_1) \rightarrow \text{spin}^c(M),$$

denoted by

$$(\sigma, L) \mapsto \sigma \otimes L,$$

which is free and transitive. Hence, up to fixing a spin^c structure, $\text{spin}^c(M)$ is isomorphic⁴ to $H^2(M; \mathbb{Z})$, cf. Example (D, 2.7). Moreover,

$$L(\sigma \otimes L) = L(\sigma) \otimes L^2 \quad \text{or, equivalently,} \quad c(\sigma \otimes L) = c(\sigma) + 2c_1(L),$$

where $c_1(L)$ is the first Chern class of L .

Proof. Let $\{U_\alpha\}$ be a good open cover of M , and let $\{g_{\alpha\beta}\}$ define the principal SO_n -bundle of M . Suppose σ is a spin^c structure, and let $\{h_{\alpha\beta}\}$ be a fitting cocycle, i.e., $\xi_0^c \circ h_{\alpha\beta} = g_{\alpha\beta}$, where ξ_0^c is the group homomorphism of (D, 3). Moreover, let L be a Hermitian line bundle and $\{\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{U}_1\}$ a set of transition functions. Then $\{h_{\alpha\beta}\lambda_{\alpha\beta}\}$ fulfills the cocycle condition and, since $\mathbb{U}_1 = \ker \xi_0^c$, also lifts the family $\{g_{\alpha\beta}\}$. Hence, it is a fitting cocycle for a spin^c structure which we denote by $\sigma \otimes L$. One readily checks that isomorphic line bundles give equivalent spin^c structure. Thus, we obtain a well-defined right action of $H^1(M, \mathbb{U}_1)$ on $\text{spin}^c(M)$.

Given another spin^c structure σ' , with fitting cocycle $\{h'_{\alpha\beta}\}$, we have

$$\xi_0^c \circ h_{\alpha\beta} = \xi_0^c \circ h'_{\alpha\beta} = g_{\alpha\beta}.$$

Since $\ker \xi_0^c = \mathbb{U}_1$, there exists a family $\{\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{U}_1\}$ such that

$$h'_{\alpha\beta} = h_{\alpha\beta}\lambda_{\alpha\beta}.$$

Clearly, $\{\lambda_{\alpha\beta}\}$ fulfills the cocycle condition (D, 7) hence defining an element in $H^1(M; \underline{\mathbb{U}}_1)$. Therefore, the action is transitive.

Now suppose that $\sigma = \sigma \otimes L$ for some line bundle L . This implies that the corresponding principal Spin_n^c -bundles are isomorphic, i.e., if $\{h_{\alpha\beta}\}$ and $\{\lambda_{\alpha\beta}\}$ are transition functions corresponding to $P_{\text{Spin}^c}(\sigma)$ and L , we can find a family $\{\Phi_\alpha : U_\alpha \rightarrow \text{Spin}_n^c\}$ such that

$$h_{\alpha\beta}\lambda_{\alpha\beta} = \Phi_\alpha h_{\alpha\beta} \Phi_\beta^{-1}.$$

⁴A set carrying a free and transitive group action is usually called a *torsor*. Thus, $\text{spin}^c(M)$ is an $H^2(M; \mathbb{Z})$ torsor.

Since $\{h_{\alpha\beta}\}$ is a fitting cocycle, the bundle map $\xi^c : P_{\text{Spin}^c}(\sigma) \rightarrow P_{\text{SO}}$ is given by the family $\{\xi_\alpha^c := \text{id} : U_\alpha \rightarrow \text{SO}_n\}$. As the same holds for $\{h_{\alpha\beta}\lambda_{\alpha\beta}\}$, commutativity of the diagram (D, 6) implies that

$$\xi_0^c \circ \Phi_\alpha = 1, \quad \text{i.e.,} \quad \Phi_\alpha : U_\alpha \rightarrow \ker \xi_0^c = U_1.$$

Since U_1 lies in the center of Spin_n^c , this shows that necessarily,

$$\lambda_{\alpha\beta} = \Phi_\alpha \Phi_\beta^{-1}.$$

Hence, L is isomorphic to the trivial line bundle. This proves that the action of $H^1(M; \underline{U}_1)$ on $\text{spin}^c(M)$ is free. \square

We now want to understand the additional structure of $\text{spin}^c(M)$ in the case of spin manifolds.

Example (D, 2.9). Let M be a spin manifold and assume that $H^2(M; \mathbb{Z})$ has no 2-torsion elements. Let ε and ε' be two spin structures on M and let $\sigma(\varepsilon)$ and $\sigma(\varepsilon')$ denote the corresponding canonical spin^c structures as defined in Example (D, 2.5). According to Proposition (D, 2.8) there exists a Hermitian line bundle L that fulfills

$$\sigma(\varepsilon') = \sigma(\varepsilon) \otimes L \quad \text{and} \quad c(\sigma(\varepsilon')) = c(\sigma(\varepsilon)) + 2c_1(L).$$

Since $c(\sigma(\varepsilon)) = c(\sigma(\varepsilon')) = 0$, this yields $2c_1(L) = 0$. Hence, according to our assumption $c_1(L) = 0$. Therefore, L is isomorphic to the trivial line bundle, i.e., $\sigma(\varepsilon)$ is equivalent to $\sigma(\varepsilon')$. This shows that all spin structures on M induce equivalent spin^c structures. We conclude that on a spin manifold M there is a canonical “origin” of $\text{spin}^c(M)$, whenever there are no 2-torsion elements in $H^2(M; \mathbb{Z})$.

Existence of spin^c structures. The interplay between the local description of principal bundles and Čech cohomology lies at the heart of understanding possible topological obstructions to the existence of spin^c structures.

The canonical group homomorphisms described in Section 1 can be assembled in the following commutative diagram, which has exact rows and

columns.

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \text{Spin}_n & \xrightarrow{\xi_0} & \text{SO}_n \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & \underline{\mathbb{U}}_1 & \longrightarrow & \text{Spin}_n^c & \xrightarrow{\xi_0^c} & \text{SO}_n \longrightarrow 1 \\
 & & \downarrow \zeta & & \downarrow \zeta^c & & \\
 & & \underline{\mathbb{U}}_1 & \equiv & \underline{\mathbb{U}}_1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & &
 \end{array}$$

The corresponding commutative diagram in Čech cohomology reads

$$\begin{array}{ccccccc}
 \vdots & & & & & & \\
 \downarrow & & & & & & \\
 H^1(M; \mathbb{Z}_2) & \longrightarrow & H^1(M; \underline{\text{Spin}}) & \longrightarrow & H^1(M; \underline{\text{SO}}) & \xrightarrow{\delta_{\xi_0}^1} & H^2(M; \mathbb{Z}_2) \\
 \downarrow & & \downarrow & & \parallel & & \downarrow \\
 H^1(M; \underline{\mathbb{U}}_1) & \longrightarrow & H^1(M; \underline{\text{Spin}}^c) & \longrightarrow & H^1(M; \underline{\text{SO}}) & \xrightarrow{\delta_{\xi_0^c}^1} & H^2(M; \underline{\mathbb{U}}_1) \\
 \downarrow & & \downarrow & & \delta_{\zeta}^1 & & \downarrow \\
 H^1(M; \underline{\mathbb{U}}_1) & \equiv & H^1(M; \underline{\mathbb{U}}_1) & & & & H^2(M; \underline{\mathbb{U}}_1) \\
 & & & & & & \downarrow \\
 & & & & & & \vdots
 \end{array} \quad (\text{D, 11})$$

Here, the δ_{\dots}^1 denote the various connecting homomorphisms. According to the preceding considerations, we can interpret a spin structure on M as an element $[h_{\alpha\beta}] \in H^1(M; \underline{\text{Spin}}_n)$ which is mapped to $[g_{\alpha\beta}]$, i.e.,

$$\xi_0[h_{\alpha\beta}] := [\xi_0 \circ h_{\alpha\beta}] = [g_{\alpha\beta}].$$

It now follows from the exactness of diagram (D, 11) that there exists a spin structure on M if and only if

$$[w_{\alpha\beta\gamma}] := \delta_{\xi_0}^1[g_{\alpha\beta}] = 0 \in H^2(M; \mathbb{Z}_2) \quad (\text{D, 12})$$

In the same way, we conclude that there exists a spin^c structure on M if and only if

$$\delta_{\xi_0^c}^1[g_{\alpha\beta}] = 0 \in H^2(M; \underline{\mathbb{U}}_1) \cong H^3(M; \mathbb{Z}), \quad (\text{D, 13})$$

where the isomorphism is the one described in Example (D, 2.7). Since $\delta_{\xi_0^c}^1[g_{\alpha\beta}]$ is the image of $[w_{\alpha\beta\gamma}]$ in $H^2(M; \underline{\mathbb{U}}_1)$, M is spin^c if and only if $[w_{\alpha\beta\gamma}]$ is mapped to 0, i.e., if and only if

$$[w_{\alpha\beta\gamma}] \in \text{Im}(\delta_\zeta^1 : H^1(M; \underline{\mathbb{U}}_1) \rightarrow H^2(M; \mathbb{Z}_2)). \quad (\text{D}, 14)$$

It is not difficult to verify that under the isomorphism $H^1(M; \underline{\mathbb{U}}_1) \cong H^2(M; \mathbb{Z})$, the connecting homomorphism δ_ζ^1 corresponds to mod 2 reduction $H^2(M; \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z}_2)$. Therefore, we can reformulate the above in the following way: *M is spin^c if and only if $[w_{\alpha\beta\gamma}]$ is the mod 2 reduction of an integral class.*

The diagram also shows that a lift of $[g_{\alpha\beta}]$ to a Spin_n^c -bundle $[h_{\alpha\beta}^c] \in H^1(M; \underline{\text{Spin}}^c)$ gives a \mathbb{U}_1 -bundle via

$$[\lambda_{\alpha\beta}] := \zeta^c[h_{\alpha\beta}^c] = [\zeta^c \circ h_{\alpha\beta}^c]$$

This \mathbb{U}_1 -bundle is the canonical line bundle of the spin^c structure (cf. Def.(D, 2.3)). By commutativity of the diagram (D, 11) we conclude that

$$\delta_\zeta^1[\lambda_{\alpha\beta}] = [w_{\alpha\beta\gamma}],$$

hence—in retrospect—condition (D, 14) for the existence of a spin^c structure σ is fulfilled by the representative of $L(\sigma)$ in $H^1(M; \underline{\mathbb{U}}_1)$.

Remark. The class $[w_{\alpha\beta\gamma}]$ which we have constructed above is a characteristic class of M depending only on the homotopy type. It is the so-called *second Stiefel-Whitney class* $w_2(M)$. The general topological construction of Stiefel-Whitney classes can be found in Milnor & Stasheff's book [40]. Moreover, it can be proved directly (cf. Lawson & Michelsohn [29], Sec. II.1) that the class $[w_{\alpha\beta\gamma}]$ satisfies the characterizing properties of the second Stiefel-Whitney class.

We summarize the above considerations in the following proposition:

Proposition (D, 2.10). *Let M be an oriented Riemannian manifold. Then M admits a spin structure if and only if its second Stiefel-Whitney class $w_2(M) \in H^2(M; \mathbb{Z}_2)$ vanishes. M admits a spin^c structure if and only if $w_2(M)$ is the mod 2 reduction of an integral class.*

Proposition (D, 2.11). *Let M be a connected, compact and oriented 3-manifold. Then M admits a spin^c structure.*

Proof. We use the condition (D, 13). Since M is connected, $H_0(M; \mathbb{Z}) = \mathbb{Z}$. Then Poincaré duality shows that $H^3(M; \mathbb{Z}) = \mathbb{Z}$ as well. In particular, $H^3(M; \mathbb{Z})$ contains no torsion elements. Therefore, the elements of $H^2(M; \mathbb{Z}_2)$ are mapped to zero in $H^3(M; \mathbb{Z})$. \square

Remark. It should be pointed out that by making use of more efficient topological methods one can prove that every compact and oriented 3-manifold is not only spin^c but even spin . Moreover, a famous result by Wu, Hirzebruch and Hopf [20] guarantees that every compact and oriented 4-manifold is spin^c .

Gauge transformations. Let (M, σ) be a spin^c manifold. An automorphism of (M, σ) is a bundle automorphism $\Phi : P_{\text{Spin}^c}(\sigma) \rightarrow P_{\text{Spin}^c}(\sigma)$ such that the corresponding diagram (D, 6) is commutative. In other words, Φ is given by a collection of smooth maps $\{\Phi_\alpha : U_\alpha \rightarrow U_1\}$ satisfying

$$h_{\alpha\beta} = \Phi_\alpha h_{\alpha\beta} \Phi_\beta^{-1},$$

where $\{h_{\alpha\beta}\}$ is a fitting cocycle for σ . Since elements of U_1 commute with elements of Spin_n^c , we conclude that

$$\Phi_\alpha|_{U_\alpha \cap U_\beta} = \Phi_\beta|_{U_\alpha \cap U_\beta}.$$

Therefore, the family $\{\Phi_\alpha\}$ defines a smooth map, say, $\gamma : M \rightarrow U_1$.

Definition (D, 2.12). The automorphism group of a spin^c manifold (M, σ) is called the *group of gauge transformations*. It is denoted by \mathcal{G} . According to the above,

$$\mathcal{G} = C^\infty(M, U_1) = H^0(M; \underline{U}_1).$$

Consider the exponential sequence⁵

$$1 \rightarrow 2\pi i\mathbb{Z} \longrightarrow \underline{i\mathbb{R}} \xrightarrow{e^x} \underline{U}_1 \rightarrow 1,$$

and let $\delta : H^0(M; \underline{U}_1) \rightarrow H^1(M; 2\pi i\mathbb{Z})$ denote the connecting homomorphism in the long exact cohomology sequence. Since $2\pi i\mathbb{Z}$ is a subsheaf of the sheaf of locally constant $i\mathbb{R}$ -valued functions⁶ on M , there exists a natural map $H^1(M; 2\pi i\mathbb{Z}) \rightarrow H^1(M; i\mathbb{R})$. According to the deRham Theorem, $H^1(M; \mathbb{R})$ is isomorphic to $H_{dR}^1(M; \mathbb{R})$, the space of closed 1-forms modulo exact 1-forms. We thus obtain a natural map

$$\rho : H^1(M; 2\pi i\mathbb{Z}) \rightarrow H_{dR}^1(M; i\mathbb{R}).$$

Using the notation $H_{dR}^1(M; 2\pi i\mathbb{Z}) := \text{im } \rho$, we have the map

$$\rho \circ \delta : H^0(M; \underline{U}_1) \rightarrow H_{dR}^1(M; 2\pi i\mathbb{Z})$$

⁵Note the difference compared to the sequence in Example (D, 2.7). The modified version is more suitable when we consider U_1 as a Lie group with Lie algebra $i\mathbb{R}$.

⁶Recall that $\underline{\mathbb{R}}$ is the sheaf of differentiable functions on M ; the sheaf of locally constant \mathbb{R} -valued functions is simply denoted by \mathbb{R} .

assigning to a gauge transformation a cohomology class of an imaginary valued, closed 1-form. Since $i\mathbb{R}$ admits partitions of unity, δ is surjective. This implies that the above composition is also surjective. Moreover,

$$\ker \rho \circ \delta = \ker \delta = \{\exp(f) \mid f \in H^0(M; i\mathbb{R})\}$$

which is essentially the identity component of the group of gauge transformations (cf. also Proposition (II, 1.3)). An explicit formula for $\rho \circ \delta$ is

$$\rho \circ \delta : H^0(M; \underline{U}_1) \rightarrow H_{dR}^1(M; 2\pi i\mathbb{Z}), \quad \gamma \mapsto [\gamma^{-1}d\gamma]. \quad (\text{D}, 15)$$

Proof. We use a description of the involved maps as it can be found, for example, in Wells' book (cf. [59], Sec. III.4). Let $\gamma \in H^0(M; \underline{U}_1)$ be a gauge transformation and let $\{U_\alpha\}$ be a good open cover of M . As U_α is contractible, $\gamma_\alpha := \gamma|_{U_\alpha}$ can be lifted to $f_\alpha : U_\alpha \rightarrow i\mathbb{R}$ via \exp . Since $\gamma_\alpha = \gamma_\beta$ on $U_\alpha \cap U_\beta$,

$$f_\beta - f_\alpha : U_\alpha \cap U_\beta \rightarrow 2\pi i\mathbb{Z} \subset i\mathbb{R}.$$

This defines a $2\pi i\mathbb{Z}$ valued Čech 1-cocycle whose cohomology class is $\delta\gamma$. On the other hand, each f_α satisfies

$$df_\alpha = \exp(-f_\alpha)d\exp(f_\alpha) = \gamma_\alpha^{-1}d\gamma_\alpha.$$

Therefore, the explicit description of the deRham isomorphism shows that $\rho[f_\beta - f_\alpha] = [\gamma^{-1}d\gamma]$. Thus, $\rho \circ \delta(\gamma) = [\gamma^{-1}d\gamma]$. \square

3 The spin^c Dirac operator

In this section we shall associate to each spin^c structure a vector bundle over M , which turns out to have a rich geometrical structure. This will give the background to introduce the spin^c Dirac operator.

Spinor bundles. Any spin^c manifold is endowed with a vector bundle carrying reflecting much of the rich geometric structure of the underlying manifold.

Definition (D, 3.1). Let (M, σ) be an n -dimensional spin^c manifold. If $\rho : \text{Spin}_n^c \rightarrow \text{GL}(\Delta)$ is a spin representation as in Definition (D, 1.2), the vector bundle associated to $P_{\text{Spin}^c}(\sigma)$ via ρ , i.e.,

$$S(\sigma) := P_{\text{Spin}^c}(\sigma) \times_\rho \Delta,$$

is called a *fundamental spinor bundle* on M . A section ψ of $S(\sigma)$ is a *spinor field* or simply a *spinor*.

The action of gauge transformations on $P_{\text{Spin}^c}(\sigma)$ induces an action on the space of spinor fields which is given by scalar multiplication⁷

$$\mathcal{G} \times C^\infty(M, S(\sigma)) \longrightarrow C^\infty(M, S(\sigma)), \quad (\gamma, \psi) \longmapsto \gamma^{-1}\psi. \quad (\text{D}, 16)$$

As a result of Proposition (D, 2.8), any other spin^c structure on M is equivalent to $\sigma \otimes L$ for some appropriate Hermitian line bundle $L \rightarrow M$. The corresponding fundamental spinor bundles are related via

$$S(\sigma \otimes L) = S(\sigma) \otimes L. \quad (\text{D}, 17)$$

Proof. Let $\{h_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Spin}_n^c\}$ be a fitting cocycle for σ and $\{\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow U_1\}$ a family transition functions for L respectively. Then $P_{\text{Spin}^c}(\sigma \otimes L)$ is defined by the cocycle $\{h_{\alpha\beta}\lambda_{\alpha\beta}\}$ and the associated vector bundle $S(\sigma \otimes L)$ is given by $\{\rho(h_{\alpha\beta}\lambda_{\alpha\beta})\}$. Since Clifford multiplication by scalars is just scalar multiplication in Δ , we obtain

$$\rho(h_{\alpha\beta}\lambda_{\alpha\beta}) = \rho(h_{\alpha\beta})\rho(\lambda_{\alpha\beta}) = \rho(h_{\alpha\beta})\lambda_{\alpha\beta}.$$

This implies the asserted formula because the right hand side defines a family of transition functions of $S(\sigma) \otimes L$. \square

Example (D, 3.2). Suppose (M, ε) is a spin manifold with spinor bundle $S(\varepsilon)$ by means of a spin representation of Spin_n . It is easy to check that this bundle equals the spinor bundle associated to the canonical spin^c structure $\sigma(\varepsilon)$ on M . In combination with Proposition (D, 2.8), the above result shows that we obtain any other spinor bundle $S(\sigma)$ over M by tensoring $S(\varepsilon)$ with a Hermitian line bundle.

The Clifford bundle. Let (M, g) be an oriented Riemannian manifold. Then we can form the *Clifford bundle* $\text{Cl}(M; g)$ over M whose fibres consist of the Clifford algebras of the tangent spaces⁸, i.e.,

$$\text{Cl}(M; g)_x = \text{Cl}(T_x M; g_x).$$

This yields a bundle of algebras which is naturally associated to the principal SO_n -bundle $P_{\text{SO}}(g)$ by means of the canonical representation

$$\text{SO}_n \rightarrow \text{Aut}(\text{Cl}_n)$$

⁷Whether one lets \mathcal{G} act via γ or γ^{-1} is merely a matter of taste. Choosing the inverse action for spinors has the advantage that \mathcal{G} must act on gauge fields (cf. (D, 24) below) in the usual way.

⁸We shall frequently identify $\text{Cl}(M; g)$ with the bundle of Clifford algebras associated to the cotangent bundle, suppressing the explicit reference to the isomorphism $TM \rightarrow T^*M$ induced by the metric.

which is given in the following way: Note that every $A \in \text{SO}_n$ gives a linear map $\mathbb{R}^n \rightarrow \text{Cl}_n$, $v \mapsto Av$ which satisfies

$$(Av)^2 = -|Av|^2 1 = -|v|^2 1.$$

Hence by the universal property of the Clifford algebra, the map $v \mapsto Av$ extends to an algebra automorphism of Cl_n .

On the other hand, each element of Spin_n^c acts on Cl_n^c by conjugation. This leaves the real subalgebra Cl_n fixed so that there is a canonical representation

$$\rho_c : \text{Spin}_n^c \rightarrow \text{Aut}(\text{Cl}_n).$$

From the definition of $\xi_0^c : \text{Spin}_n^c \rightarrow \text{SO}_n$ it is immediate that the canonical representations of SO_n and Spin_n^c on Cl_n are related in the following way:

$$\begin{array}{ccc} \text{Spin}_n^c & \xrightarrow{\rho_c} & \text{Aut}(\text{Cl}_n) \\ \xi_0^c \downarrow & & \uparrow \\ \text{SO}_n & \xrightarrow{\quad} & \end{array}$$

Therefore, if (M, σ) is a spin^c manifold, the Clifford bundle coincides with the bundle of Clifford algebras associated to $P_{\text{Spin}^c}(\sigma)$ via ρ_c .

Let Δ be a complex spinor module and $S(\sigma)$ the corresponding spinor bundle. For $g \in \text{Spin}_n^c$, $x \in \text{Cl}_n$ and $\psi \in \Delta$, we have

$$(\rho_c(g)x) \cdot (g \cdot \psi) = (g x g^{-1}) \cdot (g \cdot \psi) = (gx) \cdot \psi.$$

Since both, $S(\sigma)$ and $\text{Cl}(M)$, are bundles associated to P_{Spin^c} , the action of Cl_n on Δ thus extends to a global action, i.e., to a bundle homomorphism

$$c : \text{Cl}(M; g) \longrightarrow \text{End}(S(\sigma)).$$

Hermitian structure on $S(\sigma)$. Since a spin representation $\rho : \text{Spin}_n^c \rightarrow \text{GL}(\Delta)$ is a representation of a compact Lie group on a complex vector space, there exists a Spin_n^c -invariant Hermitian metric $\langle \cdot, \cdot \rangle$ on Δ . However, it turns out (cf. [29], Sec. I.5) that we can also achieve that

$$\langle c(x)\psi, \psi' \rangle = -\langle \psi, c(x)\psi' \rangle, \quad x \in \mathbb{R}^n, \psi, \psi' \in \Delta. \quad (\text{D}, 18)$$

Here, $c : \text{Cl}_n \rightarrow \text{End}(\Delta)$ denotes the irreducible complex representation of Cl_n which induces the spin representation.

If (M, g) is an n -dimensional oriented Riemannian manifold which admits a spin^c structure σ , then the above metric on Δ extends to a Hermitian

metric on the fundamental spinor bundle $S(\sigma) = P_{\text{Spin}^c}(\sigma) \times_{\rho} \Delta$ satisfying (D, 18) with respect to the global Clifford multiplication of vector fields (or 1-forms) on $S(\sigma)$.

Covariant derivatives on spinor bundles. We assume some familiarity with the definition of connections on principal bundles and the correspondence between them and covariant derivatives on associated vector bundles. As a general reference we refer to Kobayashi & Nomizu [26]. Moreover, a comprehensive presentation of all definitions and results we need can be found in Lawson & Michelsohn [29], Sec. II.4.

Let (M, g) be an oriented Riemannian manifold with connection 1-form $\omega = \omega^g \in \Omega^1(P_{\text{SO}}(g)) \otimes \mathfrak{so}_n$ associated to the Levi-Civita covariant derivative $\nabla = \nabla^g$ on (TM, g) . Here, \mathfrak{so}_n denotes the Lie algebra of SO_n , i.e., the real vector space of skew adjoint $(n \times n)$ -matrices endowed with the canonical Lie bracket. Suppose that $\{U_\alpha\}$ is a good open cover of M so that we may choose a section $e_\alpha = (e_1, \dots, e_n)$ of $P_{\text{SO}}(g)|_{U_\alpha}$ for each α . Then ω is determined by

$$\tilde{\omega}_{ij} := (e_\alpha^* \omega)_{ij} = g(\nabla e_i, e_j) \in \Omega^1(U_\alpha)$$

and the Levi-Civita covariant derivative over U_α is then locally given by

$$\nabla = d + \sum_{i < j} \tilde{\omega}_{ij} J_{ij}.$$

Here, $\{J_{ij}\}$ is the standard basis of \mathfrak{so}_n which is defined with respect to an orthonormal basis⁹ (e_1, \dots, e_n) of \mathbb{R}^n via $J_{ij}e_k = \delta_{ik}e_j - \delta_{jk}e_i$.

Let us now assume that in addition, M admits a spin^c structure σ .

Definition (D, 3.3). The space of connections on the principal U_1 -bundle $P_{U_1}(\sigma)$ is denoted by $\mathcal{A}(\sigma)$. An element of $\mathcal{A}(\sigma)$ is also called a *gauge field*.

$\mathcal{A}(\sigma)$ is an affine space modelled on $i\Omega^1(M)$, where we identify the Lie algebra \mathfrak{u}_1 of U_1 with the purely imaginary numbers $i\mathbb{R}$.

Let us fix $A \in \mathcal{A}(\sigma)$. Choosing local sections $s_\alpha : U_\alpha \rightarrow P_{U_1}(\sigma)|_{U_\alpha}$, we define the imaginary valued 1-forms

$$A_\alpha := s_\alpha^* A \in i\Omega^1(U_\alpha)$$

⁹Note that on the one hand, (e_1, \dots, e_n) is an orthonormal basis of \mathbb{R}^n and on the other hand, we use the same notation for the components of a local section e_α of P_{SO} . However, this ambiguity should cause no confusion since the components of $e_\alpha(x)$ form an orthonormal basis for every $x \in M$.

Since U_1 is abelian, the family $\{A_\alpha\}$ satisfies

$$A_\beta = \lambda_{\alpha\beta}^{-1} A_\alpha \lambda_{\alpha\beta} + \lambda_{\alpha\beta}^{-1} d\lambda_{\alpha\beta} = A_\alpha + \lambda_{\alpha\beta}^{-1} d\lambda_{\alpha\beta},$$

where $\{\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow U_1\}$ is the cocycle given by the sections $\{s_\alpha\}$.

The fibre product $P_{\text{SO}} \times P_{U_1}(\sigma) \rightarrow M$ is endowed with a connection $\omega \oplus A$ induced by the connection 1-forms ω and A . Let $\xi : P_{\text{Spin}^c}(\sigma) \rightarrow P_{\text{SO}} \times P_{U_1}(\sigma)$ be the two sheeted covering map encoded in the spin^c structure σ . Since ξ is equivariant with respect to

$$(\xi_0^c, \zeta^c) : \text{Spin}_n^c \rightarrow \text{SO}_n \times U_1, \quad (\text{D}, 19)$$

the product connection $\omega \oplus A$ lifts to a connection on $P_{\text{Spin}^c}(\sigma)$ via

$$\omega^A := \Phi^{-1} \circ \xi^*(\omega \times A) \in \Omega^1(P_{\text{Spin}^c}(\sigma)) \otimes \mathfrak{spin}_n^c,$$

where

$$\Phi := (\xi_0^c, \zeta^c)_* : \mathfrak{spin}_n^c \rightarrow \mathfrak{so}_n \oplus i\mathbb{R}$$

is the Lie algebra isomorphism induced by (D, 19).

Let us briefly recall an explicit description of Φ . Since there are some subtleties involved, we refer to Lawson & Michelsohn [29], Sec. I.6, for more details. As $\text{Spin}_n^c = \text{Spin}_n \times_{\mathbb{Z}_2} U_1$, there is a canonical isomorphism

$$\mathfrak{spin}_n^c \cong \mathfrak{spin}_n \oplus i\mathbb{R}. \quad (\text{D}, 20)$$

The Lie algebra \mathfrak{spin}_n can be identified with the subspace of Cl_n spanned by elements of the form $e_i e_j$ and endowed with the Lie bracket induced by the commutator in Cl_n . Then the differential of $\xi_0 : \text{Spin}_n \rightarrow \text{SO}_n$ at the unit element is given by the action of the basis in the following way

$$(\xi_0)_* : \mathfrak{spin}_n \rightarrow \mathfrak{so}_n, \quad (\xi_0)_*(e_i e_j) := 2J_{ij}.$$

The differential of $\zeta := z^2 : U_1 \rightarrow U_1$ at the unit element is

$$\zeta_* : i\mathbb{R} \rightarrow i\mathbb{R}, \quad ia \mapsto 2ia.$$

Therefore, with respect to (D, 20), the isomorphism Φ is given by

$$\Phi(e_i e_j, ia) = (2J_{ij}, 2ia). \quad (\text{D}, 21)$$

Definition (D, 3.4). The connection ω^A is called the *Clifford connection* on $P_{\text{Spin}^c}(\sigma)$ associated to A .

The section $(e_\alpha, s_\alpha) : U_\alpha \rightarrow P_{\text{SO}} \times P_{\text{U}_1}|_{U_\alpha}$ can be lifted to a section $t_\alpha : U_\alpha \rightarrow P_{\text{Spin}^c}|_{U_\alpha}$. Then it follows from (D, 21) that

$$t_\alpha^* \omega^A = \left(\frac{1}{2} \sum_{i < j} \tilde{\omega}_{ij} e_i e_j, \frac{1}{2} A_\alpha \right) \in \Omega^1(U_\alpha) \otimes \mathfrak{spin}_n^c.$$

Here, we are using the local connection 1-forms we described above.

We now consider a spin representation $\rho : \text{Spin}_n^c \rightarrow \text{U}(\Delta)$. The connection ω^A induces a covariant derivative ∇^A on the fundamental spinor bundle $S(\sigma)$. It can locally be described by

$$\nabla^A \psi = d\psi + \frac{1}{2} \sum_{i < j} \tilde{\omega}_{ij} c(e_i) c(e_j) \psi + \frac{1}{2} A_\alpha \psi. \quad (\text{D, 22})$$

Here, ψ is a spinor and c denotes Clifford multiplication. Since the representation ρ is unitary, ∇^A is compatible with the canonical metric on $S(\sigma)$, i.e.,

$$\langle \nabla^A \psi, \psi' \rangle + \langle \psi, \nabla^A \psi' \rangle = d\langle \psi, \psi' \rangle, \quad \psi, \psi' \in C^\infty(S(\sigma)).$$

Furthermore, it can be established (cf. [29], Sec. II.4.11) that ∇^A satisfies the following compatibility rule¹⁰

$$\nabla^A(c(X)\psi) = c(X)\nabla^A\psi + c(\nabla^g X)\psi, \quad (\text{D, 23})$$

where $X \in C^\infty(M, TM)$ and $\psi \in C^\infty(M, S(\sigma))$.

The group of gauge transformations $\mathcal{G} = C^\infty(M, \text{U}_1)$ acts on the set of covariant derivatives on $S(\sigma)$ by

$$(\gamma, \nabla) \longmapsto \gamma \cdot \nabla := \gamma^{-1} \nabla \gamma.$$

Then we conclude from (D, 22) that locally,

$$\begin{aligned} (\gamma \cdot \nabla^A) \psi &= \gamma^{-1} d(\gamma \psi) + \frac{1}{2} \sum_{i < j} \tilde{\omega}_{ij} c(e_i) c(e_j) \psi + \frac{1}{2} A_\alpha \psi \\ &= d\psi + \frac{1}{2} \sum_{i < j} \tilde{\omega}_{ij} c(e_i) c(e_j) \psi + \frac{1}{2} (A_\alpha + 2\gamma^{-1} d\gamma) \psi \\ &= \nabla^{A+2\gamma^{-1} d\gamma} \psi. \end{aligned}$$

Therefore, the natural action of \mathcal{G} on the space of gauge fields is given by

$$\mathcal{G} \times \mathcal{A}(\sigma) \rightarrow \mathcal{A}(\sigma), \quad \gamma \cdot A := A + 2\gamma^{-1} d\gamma. \quad (\text{D, 24})$$

¹⁰This condition can be reformulated by saying that the Clifford multiplication c is parallel with respect to ∇^A .

The spin^c Dirac operator. On a spinor bundle, we now want to construct a first-order differential operator whose square is a generalized Laplacian. This construction has a more general background. We thus briefly recall the structure which is needed to carry out the construction in general.

Definition (D, 3.5). Let $E \rightarrow M$ be a Hermitian or Euclidean vector bundle over a Riemannian manifold (M, g) . A *Dirac structure* on E is given by the following data:

- A covariant derivative ∇ on E which is compatible with the metric,
- a *Clifford structure* on E , i.e., a bundle map $c : T^*M \rightarrow \text{End}(E)$ which satisfies

$$c(\alpha) \circ c(\alpha') + c(\alpha') \circ c(\alpha) = -2g(\alpha, \alpha') \text{id}_E,$$

and which is skew adjoint with respect to the metric on E ,

- the compatibility condition

$$\nabla(c(\alpha)e) = c(\alpha)\nabla e + c(\nabla^g \alpha)e, \quad e \in C^\infty(M, E), \alpha \in \Omega^1(M).$$

If E carries a Dirac structure, it is called a *Dirac bundle* over M .

Equations (D, 18) and (D, 23) show that the canonical metric on a fundamental spinor bundle over a spin^c manifold M together with the Clifford connection satisfies all of the above conditions. We thus obtain:

Proposition (D, 3.6). *A fundamental spinor bundle $S(\sigma)$ over an oriented Riemannian spin^c manifold (M, σ) is a Dirac bundle.*

A Clifford structure yields a bundle map $c : T^*M \otimes E \rightarrow E$. Moreover, a covariant derivative on E is a \mathbb{K} -linear map $\nabla : C^\infty(E) \rightarrow C^\infty(T^*M \otimes E)$. Therefore,

$$\mathcal{D} := c \circ \nabla : C^\infty(E) \longrightarrow C^\infty(E) \tag{D, 25}$$

defines a first-order differential operator. Whenever E is a Dirac bundle over M , the operator $\mathcal{D} := c \circ \nabla$ is called a *geometric Dirac operator*.

Definition (D, 3.7). Suppose (M, σ) is a spin^c manifold. Let $A \in \mathcal{A}(\sigma)$ be a connection on the U_1 -bundle $P_{U_1}(\sigma)$. The geometric Dirac operator \mathcal{D}_A , given by the Dirac structure $(S(\sigma), \nabla^A, c)$, is called the *spin^c Dirac operator* associated to A .

Remark. If (M, ε) is a spin manifold, the spin^c Dirac operator associated to the canonical spin^c structure $\sigma(\varepsilon)$ and the flat connection on $L(\sigma(\varepsilon)) = M \times \mathbb{C}$ is the well-known spin Dirac operator. Hence, on a spin manifold, all spin^c Dirac operators are twisted versions of the spin Dirac operator.

Properties of \mathcal{D}_A . Geometric Dirac operators and, specifically, the spin^c Dirac operator have very important analytical properties some of which we state now. The corresponding proofs, which are not easy but standard calculations, can be found in any textbook on spin geometry or index theory.

Proposition (D, 3.8). (cf. [29], II.5.3). *Let E be a Dirac bundle over a Riemannian manifold M and let \mathcal{D} be the geometric Dirac operator.*

(i) *The principal symbol of \mathcal{D}^2 satisfies*

$$\sigma(\mathcal{D}^2)_\xi = -|\xi|^2, \quad \xi \in T^*M \setminus \{0\},$$

i.e., \mathcal{D}^2 is a generalized Laplacian. In particular, \mathcal{D} is an elliptic operator.

(ii) *Suppose M is oriented and compact. Then \mathcal{D} is formally self-adjoint with respect to the L^2 scalar product on $C^\infty(M, E)$.*

Proposition (D, 3.9) (Weitzenböck Formula). (cf. [29], Thm. D.12). *Suppose (M, σ) is a compact, oriented Riemannian spin^c manifold. Let \mathcal{D}_A be the spin^c Dirac operator associated to a gauge field $A \in \mathcal{A}(\sigma)$. Then*

$$\mathcal{D}_A^2 = (\nabla^A)^* \nabla^A + \frac{1}{4} s_g + \frac{1}{2} c(F_A).$$

Here, s_g denotes the scalar curvature of M and F_A is the connection 2-form of A .

Remark. Observe that F_A can be interpreted as an imaginary valued 2-form on M since U_1 is abelian. Hence, the expression $c(F_A)$ is well-defined. Recall that Clifford multiplication by k -forms is defined via the isomorphism of vector spaces $\Lambda^\bullet V \cong \text{Cl}(V)$.

Using the local description of the Clifford connection (D, 22), one straightforwardly establishes the following.

Lemma (D, 3.10). *Suppose M is an oriented Riemannian manifold equipped with a spin^c structure σ . Let $A \in \mathcal{A}(\sigma)$, and let $a \in i\Omega^1(M)$ be an imaginary valued 1-form. Then*

$$\mathcal{D}_{A+a} = \mathcal{D}_A + \frac{1}{2} c(a).$$

Remark. If M is an even dimensional oriented Riemannian manifold which admits a spin^c structure, then the fundamental spinor bundle splits into the eigenbundles of the complex volume element $\omega^c \in \text{Cl}^c(M)$. Therefore, \mathcal{D}_A decomposes into the elliptic operators

$$\mathcal{D}_A^\pm : C^\infty(M, S^\pm(\sigma)) \rightarrow C^\infty(M, S^\mp(\sigma)).$$

If M is compact, then \mathcal{D}_A^\pm are Fredholm operators. The famous Atiyah-Singer index Theorem relates $\text{ind}_{\mathbb{C}}(\mathcal{D}_A^+)$ to an integral over characteristic classes of M . However, as we are mainly interested in the three dimensional case, we will not go in more detail and refer to the literature for a further discussion.

4 Dependence on the metric

At a first glimpse the notion of a spin^c structure seems to depend on the metric g on M , which is encoded in the bundle $P_{\text{SO}}(g)$. However, it turns out that this is not the case.

The first observation is that for every Riemannian metric g on M the inclusion $P_{\text{SO}}(g) \subset P_{\text{GL}^+}$ is an equivariant homotopy equivalence, where the homotopy inverse is defined via the Gram-Schmidt orthogonalization process. Hence, for any fixed principal U_1 -bundle P_{U_1} , the equivariant two sheeted coverings of the fibre product $P_{\text{SO}}(g) \times P_{\text{U}_1}$ are in natural one-to-one correspondence with the equivariant twofold coverings of $P_{\text{GL}^+} \times P_{\text{U}_1}$. Therefore, interpreting a spin^c structure σ as a choice of principal U_1 -bundle $P_{\text{U}_1}(\sigma)$ together with a two sheeted covering of $P_{\text{GL}^+} \times P_{\text{U}_1}$ yields a possibility to define σ independently of any Riemannian metric.

Unfortunately, there is no possibility to go on in this way and construct in a metric independent way a bundle which corresponds to the spinor bundle $S(\sigma)$. The deeper reason for this is that there exists no representation of GL_n^+ stemming from some generalization of the spinor representation of SO_n (cf. Lawson & Michelsohn [29], II.5.23). We thus have to find a procedure to identify the spinor bundles $S(\sigma; g)$ and $S(\sigma; h)$ associated to different metrics g and h . The material presented here is partly taken from S. Maier's article [33] which includes an excellent summary of the results due to Bourguignon and Gauduchon in [9].

To begin with, we take a brief look on how to compare data on TM and T^*M for different metrics. Let $k : P_{\text{SO}}(g) \rightarrow P_{\text{SO}}(h)$ denote the SO -equivariant bundle map induced by $P_{\text{SO}}(g) \subset P_{\text{GL}^+} \rightarrow P_{\text{SO}}(h)$. Note that we can alternatively describe k in the following way: Let $H : TM \rightarrow TM$ be the unique positive bundle endomorphism defined by $h(.,.) = g(H.,.)$. Then

H is symmetric with respect to g , and we have the relation $k = H^{-1/2}$. The map k also gives an operation on T^*M via $k(\alpha) = \alpha \circ k^{-1}$.

We need to compare the Hodge-star-operators $*_g$ and $*_h$ associated to g and h respectively. Since for all $\alpha, \beta \in \Omega^j(M)$ we have

$$\begin{aligned} \alpha \wedge (k *_g k^{-1} \beta) &= (k^{-1} \alpha \wedge *_g k^{-1} \beta) \circ k^{-1} \\ &= g(k^{-1} \alpha, k^{-1} \beta) dv_g \circ k^{-1} = h(\alpha, \beta) dv_h, \end{aligned}$$

the result is

$$*_h = k \circ *_g \circ k^{-1}. \quad (\text{D}, 26)$$

Observe that we have used $dv_h = dv_g \circ k$ which is a consequence of the fact that k maps an orthonormal frame of (TM, g) to an orthonormal frame of (TM, h) .

In general, k need not give rise to an isometry of Hilbert spaces, $L^2(M, T^*M; g) \rightarrow L^2(M, T^*M; h)$, because $dv_g \neq dv_h$. We therefore let

$$\hat{k} := f^{-1} \cdot k,$$

where f is defined via $dv_h = f^2 dv_g$, that is, $f^2 = \det k$. We then obtain

$$\int_M h(\hat{k}\alpha, \hat{k}\beta) dv_h = \int_M f^{-2} h(k\alpha, k\beta) f^2 dv_g = \int_M g(\alpha, \beta) dv_g.$$

As a result, \hat{k} is a Hilbert space isometry $L^2(M, T^*M; g) \rightarrow L^2(M, T^*M; h)$. This gives a possibility to establish a relation between d^{*h} and d^{*g} :

$$d^{*h} = \hat{k}^2 \circ d^{*g} \circ \hat{k}^{-2}. \quad (\text{D}, 27)$$

Proof. Suppose $\alpha \in \Omega^j(M)$. Then for each $\beta \in \Omega^{j-1}(M)$, the following holds:

$$(\alpha, d\beta)_{L^2(h)} = (\hat{k}^{-2}\alpha, d\beta)_{L^2(g)} = (d^{*g}\hat{k}^{-2}\alpha, \beta)_{L^2(g)} = (\hat{k}^2 d^{*g} \hat{k}^{-2}\alpha, \beta)_{L^2(h)} \quad \square$$

Note that in contrast to (D, 26) formula (D, 27) contains derivatives of f so that explicit computations are much more involved.

We will now study the relation between $S(\sigma; g)$ and $S(\sigma; h)$. For this let $\kappa : P_{\text{Spin}^c}(\sigma; g) \rightarrow P_{\text{Spin}^c}(\sigma; h)$ be the Spin^c -equivariant bundle map induced by lifting $k \times \text{id} : P_{\text{SO}}(g) \times P_{\text{U}_1}(\sigma) \rightarrow P_{\text{SO}}(h) \times P_{\text{U}_1}(\sigma)$ to the corresponding twofold coverings. κ extends to an isometry $\kappa : S(\sigma; g) \rightarrow S(\sigma; h)$ of Hermitian vector bundles. The following is easily established.

$$\kappa(c^g(\alpha)\psi) = c^h(k(\alpha))\kappa(\psi).$$

As before, in order to obtain an isometry of Hilbert spaces $L^2(M, S; g) \rightarrow L^2(M, S; h)$ we have to define

$$\hat{\kappa} := f^{-1} \cdot \kappa.$$

Then $\hat{\kappa}$ provides a suitable instrument for pulling back the spin^c Dirac operator on $S(\sigma; h)$ to $S(\sigma; g)$. Let A be a connection on $P_{U_1}(\sigma)$. Then via the lift of the corresponding Levi-Civita connection, A gives rise to a covariant derivative $\nabla^{A; h}$ on $S(\sigma; h)$. Let \mathcal{D}_A^h denote the associated spin^c Dirac operators on $S(\sigma; h)$. Then

$$\hat{\kappa}^{-1} \circ \mathcal{D}_A^h \circ \hat{\kappa} \tag{D, 28}$$

defines a first-order elliptic operator on $S(\sigma; g)$ which is formally self-adjoint with respect to the $L^2(g)$ -metric. Fortunately, we shall not need a more explicit description of this operator. For more information concerning these points we refer to Maier [33] and Bourguignon & Gauduchon [9].

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